Group analysis of granular activated carbon filter system

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ABSTRACT

Lie symmetry method is applied to study the lifetime of granular activated carbon (GAC) filter system. Also, the optimal system of one dimensional subalgebras of the GAC filter system is constructed. The Lie invariants associated to the symmetry generators as well as the corresponding similarity reduced equations are also pointed out. Based on the obtained results, reduction of order is performed, which leads to new invariant exact solutions. The results obtained in this work are useful for understanding the dynamics of GAC filter system and can also be applied to other similar systems.

Keywords: *Lie symmetries, invariant solutions, optimal system, GAC filter system.*

1. *INTRODUCTION*

The widespread adoption of activated carbon in industrial applications stems from its remarkable capacity to extract diverse compounds from polluted water. This filtration method is also prevalent in household water purification setups. It can function independently to eliminate unpleasant tastes, odors, chlorine, and various organic substances from municipal water, thereby yielding significantly better drinking water. Additionally, it frequently serves as an initial treatment in reverse osmosis systems, mitigating the presence of numerous organic contaminants, chlorine, and other elements that could potentially damage the reverse osmosis membrane.

This paper studies the following system

$$\begin{cases} u_t + \nu u_x = -\mu w_t \\ w_t = (1-w)u - \beta w \end{cases} \tag{1}$$

Borsi, Fasano and Lenzini [1] *recently presented a system for determining the operational lifespan of granular active carbon (GAC) filters. Their method leverages a linear sorption law, allowing for straightforward computation of traveling wave solutions that describe the system's behavior. For in-depth information, consult their cited work.*

The simulation of GAC filtration plants has been the subject of various mathematical models. A pioneering effort by Bohart and Adams [2] *introduced a system of two partial differential equations to model the evolution of solute concentration and the saturation of activated sites. This early model assumed a steady-state condition for solute transport and first-order kinetics for site saturation. Since then, this fundamental model has been significantly enhanced through the integration of numerous additional features. (See [3] and the reference therein).*

Lie group theory has emerged as a fundamental framework in the analysis of differential equations, as extensively documented in the literature [4, 5, 8]. *A significant benefit of identifying the symmetry group of a differential equation system is the capacity to generate new solutions from known ones. The Lie group method is widely acknowledged for its effective and direct approach to obtaining exact solutions for nonlinear differential equations. Moreover, it provides a basis for deriving various other types of exact*

solutions for partial differential equations, such as traveling wave, soliton, and fundamental solutions [6, 7]. *Additional applications of this theory within academic discourse include the reduction of order in ordinary differential equations, the systematic construction of invariant solutions, and the identification of appropriate linearizing transformations.*

2. *Lie symmetries of the GAC filter system*

In this section, the classical Lie symmetry method for the GAC system has been performed. consider the one parameter Lie group of infinitesimal transformations with independent variables x, t and dependent variables u, w,

$$\begin{aligned}\tilde{x} &= x + s\xi(x,t,u,w) + O(s^2)\\ \tilde{t} &= t + s\eta(x,t,u,w) + O(s^2)\\ \tilde{u} &= u + s\varphi(x,t,u,w) + O(s^2)\\ \tilde{w} &= w + s\psi(x,t,u,w) + O(s^2)\end{aligned} \tag{2}$$

where s is the group parameter and ξ, η, φ, and ψ are the infinitesimals of the transformations for problem. The associated vector field is of the form:

$$\mathbf{X} = \xi(x,t,u,w)\partial_x + \eta(x,t,u,w)\partial_t + \varphi(x,t,u,w)\partial_u + \psi(x,t,u,w)\partial_w \tag{3}$$

and, its first prolongation is

$$\mathrm{Pr}^{(1)}\mathbf{X} = \mathbf{X} + \varphi^x\partial_{u_x} + \varphi^t\partial_{u_t} + \psi^x\partial_{w_x} + \psi^t\partial_{w_t} \tag{4}$$

where

$$\begin{aligned}\varphi^x &= D_x(\varphi - \xi u_x - \eta u_t) + \xi u_{x^2} + \eta u_{xt}\\ \varphi^t &= D_t(\varphi - \xi u_x - \eta u_t) + \xi u_{xt} + \eta u_{t^2}\\ \psi^x &= D_x(\psi - \xi w_x - \eta w_t) + \xi w_{x^2} + \eta w_{xt}\\ \psi^t &= D_t(\psi - \xi w_x - \eta w_t) + \xi w_{xt} + \eta w_{t^2}\end{aligned} \tag{5}$$

By theorem (6.5) in [4] the vector field ***X*** *generates a one parameter symmetry group of GAC system if and only if*

$$\begin{cases}\mathrm{Pr}^{(1)}\mathbf{X}[u_t + \nu u_x + \mu w_t] = 0,\\ \mathrm{Pr}^{(1)}\mathbf{X}[w_t - (1-w)u + \beta w] = 0,\end{cases} (mod \quad Eq(1)) \tag{6}$$

The condition (6) *is equivalent to*

$$\begin{cases}\varphi^t + \nu\varphi^x + \mu\psi^t = 0,\\ \psi^t - \varphi + u\psi + w\varphi + \beta\psi = 0,\end{cases} (mod \quad Eq(1)) \tag{7}$$

Substituting (5) *into* (7)*, and equating the coefficients of the various monomials in partial derivatives with respect to x and various power of u, we can find the determining equations for the symmetry group of the system* (1)*. Solving these equations, we get the following forms of the coefficient functions*

$$\xi = c_1 x + c_2, \quad \eta = -c_1 t + \frac{2c_1}{\nu} x + c_3, \qquad \varphi = c_1(\beta + u), \qquad \psi = c_1(1-w) \tag{8}$$

where c_1, c_2 and $c_3$are arbitrary constant. Thus, the Lie algebra g of infinitesimal symmetry of the GAC system is spanned bye the three vector fields

$$\mathbf{X}_1 = \partial_x, \qquad \mathbf{X}_2 = \partial_t, \qquad \mathbf{X}_3 = \nu x \partial_x + (2x - t\nu)\partial_t + \nu(u+\beta)\partial_u + \nu(1-w)\partial_w \tag{9}$$

The commutation relations between these vector fields are given in the Table 1.

Table 1
The commutator table

$[\mathbf{X}_i, \mathbf{X}_j]$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$
$\mathbf{X}_1$	0	0	$\nu\mathbf{X}_1 + 2\mathbf{X}_2$
$\mathbf{X}_2$	0	0	$-\nu\mathbf{X}_2$
$\mathbf{X}_3$	$-\nu\mathbf{X}_1 - 2\mathbf{X}_2$	$\nu\mathbf{X}_2$	0

*The Lie algebra **g** is solvable, because if $g^{(1)} = \langle X_i, [X_i, X_j]\rangle = [g, g]$, then we have $g^{(1)} = \langle X_1, X_2, X_3\rangle$ and $g^{(2)} = [g^{(1)}, g^{(1)}] = \langle \upsilon X_1 + 2X_2, -\upsilon X_2\rangle$, so we have a chain of ideals$g^{(1)} \supset g^{(2)} \supset 0$.*

*Here **g** has no non-trivial Levi decomposition because **g** has no non-trivial radical, i.e., if **r** be the radical of **g**, then **g** = **r**. Also, **g** is not semi simple, because its killing form is degenerate.*

To obtain the group transformation which is generated by the infinitesimal generators X_i for i=1,2,3 we need to solve the three systems of first order ordinary differential equations

$$\begin{aligned}
\frac{d\tilde{x}(s)}{ds} &= \xi_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s), \tilde{w}(s)), \quad \tilde{x}(0) = x \\
\frac{d\tilde{t}(s)}{ds} &= \eta_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s), \tilde{w}(s)), \quad \tilde{t}(0) = t \\
\frac{d\tilde{u}(s)}{ds} &= \varphi_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s), \tilde{w}(s)), \quad \tilde{u}(0) = u \\
\frac{d\tilde{w}(s)}{ds} &= \psi_i(\tilde{x}(s), \tilde{t}(s), \tilde{u}(s), \tilde{w}(s)), \quad \tilde{w}(0) = w
\end{aligned}$$

Exponentiating the infinitesimal symmetries of system (1)*, we get the one-parameter groups G_i generated by X_i for i=1,2,3.*

Lemma 2.1 *The one-parameter groups $G_i(s): M \to M$ generated by X_i for i=1,2,3 are given by*

$$\begin{aligned}
G_1 &: (t, x, u, w) \mapsto (x + s, t, u, w) \\
G_2 &: (t, x, u, w) \mapsto (x, t + s, u, w) \\
G_3 &: (t, x, u, w) \mapsto \left(x\mathrm{e}^{\nu s}, \frac{x\mathrm{e}^{\nu s} + (t\nu - x)\mathrm{e}^{-\nu s}}{\nu}, \mathrm{e}^{\nu s}(u+\beta) - \beta, 1 + \mathrm{e}^{-\nu s}(w-1) \right).
\end{aligned}$$

Theorem 2.2 *if u = f (x; t) and w = g (x; t) are solutions of GAC system, so are the functions*

$$G_1(s)\cdot f(x,t)=f(x-s,t)$$
$$G_1(s)\cdot g(x,t)=g(x-s,t)$$
$$G_2(s)\cdot f(x,t)=f(x,t-s)$$
$$G_2(s)\cdot g(x,t)=g(x,t-s)$$
$$G_3(s)\cdot f(x,t)=\beta(\mathrm{e}^{\nu s}-1)+f\left(x\mathrm{e}^{-\nu s},\frac{\mathrm{e}^{\nu s}}{\nu}(t\nu-x+\mathrm{e}^{-2\nu s}x)\right)\mathrm{e}^{\nu s}$$
$$G_3(s)\cdot g(x,t)=1-\mathrm{e}^{-\nu s}+g\left(x\mathrm{e}^{-\nu s},\frac{\mathrm{e}^{\nu s}}{\nu}(t\nu-x+\mathrm{e}^{-2\nu s}x)\right)\mathrm{e}^{-\nu s}.$$

3. *Optimal system of the GAC filter system*

In general, each s-parameter subgroup H within the complete symmetry group G of a differential equation system with p>s independent variables corresponds to a family of group-invariant solutions. Due to the generally infinite number of such subgroups, enumerating all potential group-invariant solutions is often impractical. Therefore, an effective and systematic method for classifying these solutions is necessary, leading to an 'optimal system' of group-invariant solutions from which all other such solutions can be derived.

By theorem (3.2) in [4]*, determining an optimal system of subgroups is equivalent to identifying an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem closely aligns with classifying the orbits of the adjoint representation, given that each one-dimensional subalgebra is defined by a non-zero vector within the Lie algebra* ***g****. The conventional approach to this problem involves simplifying a general element V in* ***g*** *through successive adjoint transformations. Thus, we will deal with the construction of the optimal system of subalgebras of* ***g****.*

To compute the adjoint representation, we use the Lie series

$$Ad(\exp(\varepsilon\mathbf{X}_i)\mathbf{X}_j)=\mathbf{X}_j-\varepsilon[\mathbf{X}_i,\mathbf{X}_j]+\frac{\varepsilon^2}{2}[\mathbf{X}_i,[\mathbf{X}_i,\mathbf{X}_j]]-\cdots \qquad (10)$$

Where $[X_i,X_j]$ *is the commutator for the Lie algebra,*ε *is a parameter, and i,j=1,2,3. Then we have the Table 2.*

Table 2
Adjoint representation table.

Ad	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$
$\mathbf{X}_1$	0	0	$\mathbf{X}_3-\varepsilon(\nu\mathbf{X}_1+2\mathbf{X}_2)$
$\mathbf{X}_2$	0	0	$\mathbf{X}_2+\varepsilon\nu\mathbf{X}_2$
$\mathbf{X}_3$	$\mathbf{X}_1+\varepsilon(\nu\mathbf{X}_1+2\mathbf{X}_2)$	$\mathbf{X}_2-\varepsilon\nu\mathbf{X}_2$	0

Theorem 3.1 *An optimal system of one-dimensional Lie algebras of the GAC system is provided by*

$$(i)\ a_1X_1+X_3, \qquad (ii)\ a_1X_1+X_2$$

proof: *Consider the symmetry algebra* $\mathbf{g}$ *of the system* (1) *whose adjoint representation was determined in table 2 and let* $F_i^s: g \to g$ *defined by* $X \mapsto Ad(\exp(\varepsilon X_i)X)$ *is a linear map, for i=1,2,3. The matrices* M_i^ε *and* F_i^ε, *with respect to basis* $\{X_1, X_2, X_3\}$ *are*

$$M_1^{\varepsilon}=\begin{pmatrix}1&0&\varepsilon\nu\\0&1&2\varepsilon\\0&0&1\end{pmatrix},\qquad M_2^{\varepsilon}=\begin{pmatrix}1&0&0\\0&1&-\varepsilon\nu\\0&0&1\end{pmatrix},\qquad M_3^{\varepsilon}=\begin{pmatrix}e^{-\varepsilon\nu}&0&0\\\frac{e^{-\varepsilon\nu}-e^{\varepsilon\nu}}{\nu}&e^{\varepsilon\nu}&0\\0&0&1\end{pmatrix}$$

Let $V=\sum_{i=1}^{3}a_iX_i$ *is a nonzero vector field in* $\mathbf{g}$ *for some constants* a_i. *We will simplify as many of the coefficients* a_i *as possible by acting these matrices on a vector field V alternatively.*
Suppose first that $a_3 \neq 0$, *scaling* **V,** *if necessary, we can assume that* $a_3=1$, *then we can make the coefficient of* X_2 *vanish by acting* M_2^ε *and setting* $\varepsilon=\frac{a_2}{\nu}$ *so* **V** *reduced to case* (i).
If $a_3=0$ *and* $a_2 \neq 0$, *then we can not make vanish the coefficients of* X_1 *by acting any matrices* M_i^ε. *Scaling* **V,** *if necessary, we can assume that* $a_2=1$ *and* **V** *reduced to second case.* □

4. *Symmetry reduction and similarity solutions*

The symmetry group method will be applied to equation (1) to establish a direct connection to a system of ordinary differential equations. This involves identifying specific linear combinations of infinitesimals and their corresponding invariants. Since system (1) is expressed in coordinates (x, t, u, w), its reduction necessitates transforming it into a simplified form within a new coordinate system. These new coordinates (y, U, W) will be constructed from the independent invariants corresponding to the infinitesimal generator. Subsequently, applying the chain rule allows for the derivation of the reduced equation in these new coordinates. This process yields invariant solutions with respect to the identified symmetries. (for more details see [4,9])

Case 1. *For the operator* X_3 *characteristic equation is*

$$\frac{dt}{2x-t\nu}=\frac{dx}{\nu x}=\frac{du}{\nu(u+\beta)}=\frac{dw}{\nu(1-w)}. \tag{11}$$

The corresponding invariants are

$$y=t-\frac{x^2+c\nu}{\nu x},\qquad U=u-\beta-cx,\qquad W=w-1-\frac{c}{x}. \tag{12}$$

The derivatives of u and w are

$$u_t=U_y,\qquad w_t=W_y,\qquad u_x=\frac{c\nu-x^2}{\nu x^2}U_y-c. \tag{13}$$

Substituting (13) *into the system* (1), *we obtain the ordinary differential equation system*

$$\begin{cases} \frac{c\nu}{x^2}U_y - c\nu = -\mu W_y \\ W_y = (\frac{c}{x} - W)(U + \beta - \frac{c}{x}) - \beta(W - 1 - \frac{c}{x}) \end{cases}$$

Case 2. *The invariants of X_1 are y=t, $U = u$ and $W = w$ so the reduced form of the system* (1) *is*

$$\begin{cases} U_y = -\mu W_y \\ W_y = (1-W)U - \beta W \end{cases}$$

solution of this system is

$$u = \frac{c_1 - \beta}{2} + \frac{1}{2}\tanh\left((t + c_2)\frac{\sqrt{\delta}}{2}\right)\sqrt{\delta},$$

$$w = \frac{\frac{1}{4}\left(1 - \tanh((t + c_2)\frac{\sqrt{\delta}}{2})^2\right)\delta + \mu u}{\mu(u + \beta)}$$

where c_1, c_2 are arbitrary constant and $\delta = 4\mu\beta + 2c_1\beta + \beta^2 + c_1^2$.

Case 3. *The invariants of $\alpha X_1 + X_2$ are $y = x - \alpha t$, $U = u$ and $W = w$ so the reduced form of the system* (1) *is*

$$\begin{cases} (\nu - \alpha)U_y = \alpha\mu W_y \\ -\alpha W_y = (1-W)U - \beta W \end{cases}$$

solution of this system is

$$u = \frac{\alpha\beta - \nu\beta + c_1\alpha^2 - c_1\alpha\nu + \tan\left(\frac{\sqrt{\delta}(y + c_2)}{2\alpha(\nu - \alpha)}\right)\sqrt{\delta}}{2(\nu - \alpha)},$$

$$w = \frac{4\mu u\alpha\nu\cos\left(\frac{\sqrt{\delta}(y + c_2)}{2\alpha(\nu - \alpha)}\right)^2 - 4\mu u\alpha^2\cosh\left(\frac{\sqrt{\delta}(y + c_2)}{2\alpha(\nu - \alpha)}\right)^2 + \delta}{4\alpha\mu(\beta + u)(\nu - \alpha)\cos\left(\frac{\sqrt{\delta}(y + c_2)}{2\alpha(\nu - \alpha)}\right)^2}$$

where c_1, c_2 are arbitrary constant and in this case

$$\delta = 4\alpha\mu\beta\nu - 4\alpha^2\mu\beta + 2\alpha c_1\nu^2\beta - 4\alpha^2 c_1\nu\beta + 2\alpha^3 c_1\beta$$
$$-\nu^2\beta^2 + 2\nu\beta^2\alpha - \alpha^2\beta^2 - \alpha^2 c_1^2\nu^2 + 2\alpha^3 c_1^2\nu - \alpha^4 c_1^2$$

Case 4. *The invariants of the invariants of* X_2 *are* $y = x$, $U = u$ *and* $W = w$ *so the reduced form of the system* (1) *is*

$$\begin{cases} \nu U_x = 0 \\ (1-W)U - \beta W = 0 \end{cases}$$

solution of this equation is $u = c_1$ *and* $w = \frac{c_1}{c_1+\beta}$.

5. Conclusion

In this paper by applying the criterion of invariance of the equation under the infinitesimal prolonged infinitesimal generators, we find the most general Lie point symmetries group of the GAC system. The Lie algebra of symmetries of the GAC system, is discussed. It is proved that $\boldsymbol{g}$ *is a solvable non-semisimple Lie algebra and has no Levi decomposition. All the group invariant solutions and the symmetry reduced equations associated to infinitesimal symmetry generators are computed and find some similarity solutions. We have constructed the optimal system of one-dimensional subalgebras of GAC system. The latter, creates the preliminary classification of group invariant solutions.*

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