Differential invariants and conservation lows of granular activated carbon filter system

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ABSTRACT

In this study, differential invariants of the granular activated carbon (GAC) filter system are investigated. Investigation of differential invariants offers a deeper understanding of the system's inherent properties that remain unchanged under specific transformations. Nonclassical symmetry method is applied to study the lifetime of (GAC) filter system. Finally, conservation laws of the GAC system are presented. The derivation of conservation laws further enhances the analysis by providing conserved quantities that remain constant over time.

Keywords: *nonclassical symmetries, differential invariants, conservation laws, GAC filter system.*

1. *INTRODUCTION*

Activated carbon is extensively employed in industrial settings due to its notable efficacy in removing a wide array of compounds from contaminated water. This filtration technique is also commonly integrated into domestic water purification systems. It independently removes undesirable tastes, odors, chlorine, and various organic compounds from municipal water, thereby enhancing potable water quality. Furthermore, activated carbon frequently functions as a pre-treatment stage in reverse osmosis systems, mitigating the impact of numerous organic pollutants, chlorine, and other constituents that could potentially compromise membrane integrity. This paper studies the following system

$$\begin{cases} u_t + \nu u_x = -\mu w_t \\ w_t = (1-w)u - \beta w \end{cases} \tag{1}$$

Borsi, Fasano, and Lenzini [1] *recently introduced a methodology for assessing the operational lifespan of granular activated carbon (GAC) filters. Their approach utilizes a linear sorption law, which facilitates the direct calculation of traveling wave solutions that characterize the system's dynamics. Further details can be found in their referenced publication.*

Various mathematical models have been developed to simulate granular activated carbon (GAC) filtration plants. A seminal contribution by Bohart and Adams [2] *established a two-partial differential equation system to describe the evolution of solute concentration and the saturation of activated sites. This foundational model posited steady-state solute transport and first-order kinetics for site saturation. Subsequently, this core model has undergone substantial refinements through the incorporation of numerous additional complexities (refer to* [3] *and its cited literature).*

Classical symmetry method for (GAC) filter system (1) *studied in* [4] *and find symmetry group of* (1). *Differential invariants help us to find general systems of differential equations which admit a prescribed symmetry group. One say, if G is a symmetry group for a system of PDEs with functionally differential invariants, then, the system can be rewritten in terms of differential invariants.*

The nonclassical symmetry method of reduction was devised originally by Bluman and Cole in [5], *to find new exact solutions of the heat equation. The description of the method is presented in* [6,7]. *A new procedure for finding nonclassical symmetries have been proposed by Bila and Niesen in* [8].

In applied sciences and engineering, numerous Partial Differential Equations (PDEs) are formulated as continuity equations, representing the conservation of fundamental quantities such as mass, momentum, energy, or electric charge. These equations are prevalent across various fields, including fluid mechanics, particle and quantum physics, plasma physics, elasticity, gas dynamics, electromagnetism, magneto-hydrodynamics, and nonlinear optics. Within the study of PDEs, conservation laws play a crucial role in examining integrability and linearization mappings, as well as in proving the existence and uniqueness of solutions. Furthermore, they are instrumental in analyzing the stability and global behavior of these solutions [9,10,12,13].

2. *Characterization of differential invariants*

Differential invariants facilitate the formulation of general systems of differential equations that exhibit a predetermined symmetry group. Specifically, if a system of Partial Differential Equations (PDEs) possesses a symmetry group G with functionally independent differential invariants, the system can then be re-expressed using these differential invariants.

Symmetry group of system (1) *which calculated in* [4] *is*

$$\mathbf{X}_1 = \partial_x, \qquad \mathbf{X}_2 = \partial_t, \qquad \mathbf{X}_3 = \nu x \partial_x + (2x - t\nu)\partial_t + \nu(u+\beta)\partial_u + \nu(1-w)\partial_w$$

For finding the differential invariants of the system (1) *up to order 2, we should solve the following systems of PDEs:*

$$\frac{\partial I}{\partial x}, \quad \frac{\partial I}{\partial t}, \quad \nu\frac{\partial I}{\partial x} + (2x - t\nu)\frac{\partial I}{\partial t} + \nu(\beta + u)\frac{\partial I}{\partial u} + \nu(1-w)\frac{\partial I}{\partial w}, \tag{2}$$

where I *is a smooth function of* (x,t,u,w)

$$\frac{\partial I_1}{\partial x}, \quad \frac{\partial I_1}{\partial t}, \quad \nu\frac{\partial I_1}{\partial x} + \cdots - 2u_t\frac{\partial I_1}{\partial u_x} + 2\nu u_t\frac{\partial I_1}{\partial u_t} - 2(\nu w_x + w_t)\frac{\partial I_1}{\partial w_x}, \tag{3}$$

where I_1 *is a smooth function of* $(x,t,u,w,u_x,u_t,w_x,w_t)$

$$\frac{\partial I_2}{\partial x}, \quad \frac{\partial I_2}{\partial t}, \quad \nu\frac{\partial I_2}{\partial x} + \cdots - (\nu w_{xt} + 2w_{tt})\frac{\partial I_2}{\partial w_{xt}} + \nu w_{tt}\frac{\partial I_2}{\partial w_{tt}}, \tag{4}$$

where I_2 *is a smooth function of* $(x,t,u,w,u_x,u_t,u_{xx},u_{xt},u_{tt},w_x,w_t,w_{xx},w_{xt},w_{tt})$. *The solution of PDEs systems* (2), (3) *and* (4) *coming in table 1, where * and ** are refer to ordinary invariants and first order differential invariants respectively*

Table 1
Differential invariants table.

vector field	ordinary invariant	1st order	2nd order
$\mathbf{X}_1$	t, u, w	$*, u_x, u_t, w_x, w_t$	$*, **, u_{xx}, u_{xt}, u_{tt}, w_{xx}, w_{xt}, w_{tt}$
$\mathbf{X}_2$	x, u, w	$*, u_x, u_t, w_x, w_t$	$*, **, u_{xx}, u_{xt}, u_{tt}, w_{xx}, w_{xt}, w_{tt}$
$\mathbf{X}_3$	$\frac{(t\nu-x)x}{\nu}, \frac{u+\beta}{x}, (w-1)x$	$*, \frac{u_t}{x^2}, \frac{u_t+\nu u_x}{\nu}, w_t, \frac{(w_t+\nu w_x)x^2}{\nu}$	$*, **, \frac{u_{tt}}{x^3}, \frac{u_{tt}+\nu u_{xt}}{\nu x}, \frac{(u_{tt}+2\nu u_{xt}+\nu^2 u_{xx})x}{\nu^2}, \frac{w_{tt}}{x}, \frac{(w_{tt}+\nu w_{xt})x}{\nu}, \frac{x^3(w_{tt}+2\nu w_{xt}+w_{xx}\nu^2)}{\nu^2}$

3. *Investigating of nonclassical symmetries*

In this section we would like to apply the nonclassical method to the system (1). *The nonclassical symmetry method has become the focus of a lot of research and many applications to physically important partial differential equations as in* [8]. *For the nonclassical method, we must add the invariance surface condition to the given equation, and then apply the classical symmetry method. This can also be conveniently written as:*

$$\mathrm{Pr}^{(n)}\mathbf{X}(\Delta_\ell)|_{E\cap E_Q^{(n)}}=0, \qquad \ell=1,\cdots,m \tag{5}$$

where E is a nonlinear system of partial differential equations of order n in p independent and q dependent variables is given as a system of equations

$$\Delta_\ell(x,u^{(n)})=0, \qquad \ell=1,\cdots,m \tag{6}$$

involving $x=(x^1,\cdots,x^p)$, $u=(u^1,\cdots,u^q)$*and the derivatives of u with respect to x up to n, where* $u^{(n)}$*represents all the derivatives of u of all orders from 0 to n and* E_Q *is the system of partial differential equations*

$$Q^\alpha(x,u,u^{(1)})=\varphi^{(\alpha)}(x,u)-\sum_{i=1}^{p}\xi^i(x,u)u_i^\alpha=0, \qquad \alpha=1,\cdots,q \tag{7}$$

known as the invariant surface conditions. The q-tuple $Q=(Q^1,\cdots,Q^q)$ *is known as the characteristic of the following vector field*

$$\mathbf{X}=\xi(x,t,u,w)\partial_x+\eta(x,t,u,w)\partial_t+\varphi(x,t,u,w)\partial_u+\psi(x,t,u,w)\partial_w \tag{8}$$

Without loss of generality, we choose $\eta=1$ *of the vector field* (8). *In this case using* E_Q *we have*

$$u_t+\xi u_x=\varphi, \qquad w_t+\xi w_x=\psi \tag{9}$$

By reducing the initial system using the invariant surface condition (9), *the equivalent form of system* (1) *is obtained. Invariance of the equivalent equation under a Lie group of point transformations, with infinitesimal generator* (8) *leads to the determining equations. Substituting* $\eta = 1$ *into the determining equations we obtain the determining equations of the nonclassical symmetries of the original system* (1). *Solving the system obtained by this procedure, the only solutions we found were exactly the solution obtained through the classical symmetry approach. This means that no supplementary symmetries of nonclassical type are specific for the GAC filter system.*

4. *Conservation law of the GAC filter system*

Various methods have been developed to address conservation laws. These include, but are not limited to, approaches based on Noether's theorem, the multiplier method, utilizing the relationship between a PDE's conserved vector and its Lie-Bäcklund symmetry generators, and direct methods, etc. [10, 11, 12].

Definition 4.1 *A local conservation law of the PDE system* (6) *is a divergence expression*

$$D_i\Phi^i[u] = D_1\Phi^1[u] + \cdots + D_n\Phi^n[u] = 0 \tag{10}$$

holding for all solutions of the system (6). *In* (10), $\Phi^i[u] = \Phi^i(x, u, \partial_u, \cdots, \partial_u^r)$, $i = 1, \cdots, n$, *are called fluxes of the conservation law, and the highest-order derivative* (r) *present in the fluxes* $\Phi^i[u]$ *is called the order of a conservation law.* [10]

Specifically, a set of multipliers $\{\Lambda_\ell[U]\}_{\ell=1}^m = \{\Lambda_\nu(x, U, \partial_U, \cdots, \partial_U^r)\}_{\ell=1}^m$ *can lead to a divergence expression for system* $\Delta_\ell(x, u^{(n)})$ *if identity*

$$\Lambda_\ell[U]\Delta_\ell[U] \equiv D_i\Phi^i[U] \tag{11}$$

holds identically for arbitrary functions $U(x)$. *Then for solutions* $U(x) = u(x)$ *of system* (6), *if* $\Lambda_\ell[U]$ *is non-singular, one has local conservation law* $\Lambda_\ell[u]\Delta_\ell[u] = D_i\Phi^i[u] = 0$.

Definition 4.2 *The Euler operator with respect to* U^j *for* $j = 1, \cdots, q$ *is the operator defined by*

$$E_{U^j} = \frac{\partial}{\partial U^j} - D_i\frac{\partial}{\partial U^j} + \cdots + (-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial U^j_{i_1\cdots i_s}} + \cdots \tag{12}$$

Theorem 4.3 A *set of non-singular local multipliers* $\{\Lambda_\ell(x, U, \partial_U, \cdots, \partial_U^r)\}_{\ell=1}^m$ *yields a local conservation law for the system* $\Delta_\ell(x, u^{(n)})$ *if and only if the set of identities*

$$E_{U^j}(\Lambda_\ell(x, U, \partial_U, \cdots, \partial_U^r)\Delta_\ell(x, u^{(n)})) \equiv 0, \qquad j = 1, \cdots q \tag{13}$$

holds for arbitrary functions $U(x)$. (*Theorem 1.3.3* [10])

The system of equations (13) *provides a set of linear determining equations used to identify all possible local conservation law multipliers for system* (6).

Now, we seek all local conservation law multipliers of the form $\Lambda_1 = \xi(x,t,u,w)$ *and* $\Lambda_2 = \phi(x,t,u,w)$ *of system* (1). *The determining equations* (13) *become*

$$E_U[\xi(x,t,U,W)(U_t + \nu U_x + \mu W_t) + \phi(x,t,U,W)(W_t - (1-W)U + \beta W)] \equiv 0 \quad (14)$$

where $U(x,t)$ *and* $W(x,t)$ *are arbitrary functions. Equation* (14) *split with respect to first order derivatives of* U *and* W *to yield the determining PDE system whose solutions are the sets of local multipliers of all nontrivial local conservation laws.*

The solution of the determining system (14) *given by* $\phi = 0$ *and* $\xi = c_1$ *where* c_1 *is arbitrary constant, so local multipliers given by*$(\xi,\phi) = (1,0)$.

Local multiplier (ξ,ϕ) *determines a nontrivial local conservation law* $D_t\Psi + D_x\Phi = 0$ *with the characteristic form*

$$D_t\Psi + D_x\Phi \equiv \xi(U_t + \nu U_x + \mu W_t) + \phi(W_t - (1-W)U + \beta \quad (15)$$

so, we have conservation low of the GAC filter system respect to multiplier $(\xi,\phi) = (1,0)$

$$D_t(u + \mu w) + D_x(\nu u) = 0.$$

5. *Conclusion*

In this study, we rigorously applied the nonclassical symmetry method to the GAC filter model. Our analysis revealed that, contrary to some systems, the GAC filter model does not admit any additional nonclassical symmetries beyond those inherent to its structure. Complementing this, we meticulously computed the differential invariants associated with the model, which are crucial for understanding its underlying structure. Subsequently, leveraging the robust multiplier method, we successfully established a significant conservation law for the system, providing valuable insights into its dynamic behavior and conserved quantities.

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