Defining the optimal length of the warranty period in a new warranty policy with constraints

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ABSTRACT

The main goal of this research is to provide an optimal warranty length under the constraints and using linear pro rata warranty. For this purpose, a coherent system consisting of parts that are connected in parallel is considered. For the said system, limitations such as the number of parts, available space, and minimum reliability are considered. Finally, the average and total cost functions were calculated during the initial performance test and warranty periods. Using a numerical example, graphs of total and average cost function and optimal length of warranty were explained.

Keywords: ***Optimal length, Warranty policies, Linear pro rata warranty, Burn-in test, Minimal and general repair***

1. INTRODUCTION

The increasing competition among suppliers has led consumers to have a wide range of choices, enabling them to select the best option from various sources. One of the main factors influencing consumer choice, especially when products are relatively similar (with negligible price differences, similar performance, etc.), is the presence of attractive warranty/guaranty offers [1]. A warranty or guaranty is essentially a contract between the manufacturer and the buyer, accompanying the sale of a product, which specifies the type and extent of compensation for product failure during the warranty period [2]. Fair warranty policies can provide consumers with confidence in their choice and reduce the potential risks associated with their decision.

The presence of warranties, especially for products which newly introduced to the market, is a powerful tool for gaining market share against established and capable competitors. Therefore, suppliers are always striving to offer attractive warranties to consumers of their products, which, while gaining customer trust and satisfaction, can also meet their required profit margins. To this end, various warranty policies have been defined by suppliers in recent years. Different proposals for offering warranties are referred to as warranty policies [3]. A warranty policy refers to the type of service offered to the customer and the manner in which it is provided. When offering warranties, manufacturers consider several key variables. These variables include the type of warranty or repair method, the manner of providing warranty services (in terms of costs), and the duration of the warranty. Among the various types of warranties are one-dimensional or two-dimensional warranties, renewable or non-renewable warranties, and replaceable warranties. In parallel with the definition of various policies, research teams have also sought to evaluate and analyze them under different conditions, estimating the benefits for both suppliers and consumers [4].

In today's competitive market, manufacturers use various tools to encourage customers to use their services or purchase their products. One such tool is the implementation of different warranty policies during product sales and offering them to customers. Providing warranty services entails costs for the manufacturer.

To succeed in the market and offer competitive pricing, it is essential to keep these costs to a minimum (an optimal value) while simultaneously providing an appropriate warranty policy. Therefore, accurately calculating the costs associated with the proposed warranty policy is crucial.

In the present study, several repairable, identical components with the same lifespan function are arranged in parallel to enhance the system's reliability (achieving the minimum acceptable reliability). To prevent early failures in the product, the system undergoes initial performance testing (namely, burn-in test) before being released to the market. This allows for the early detection of any defects, preventing defective products from being sent to the market. During the initial performance testing, if a failure occurs in the product, depending on the type of failure (minor failure with probability p_1 or major failure with probability $1-p_1$), the product may either undergo minor repairs or be replaced entirely (replacement). If a system successfully passes this initial testing phase, the product is released to the market along with a warranty offer of length W. During the warranty period, if the product fails, depending on the type of failure (minor failure with probability p_2 or major failure with probability $1-p_2$), it will be repaired either partially (minor repair) or generally (general repair). According to a linear cost-sharing function (pro rata function), a portion of the repair cost is borne by the manufacturer.

A notable aspect of this model is that not all failures occurring during the warranty period are covered under the warranty services. For example, if a failure results from mechanical impact on the product, it is not covered under the warranty. On average, it is assumed that p percent of warranty claims are covered under the warranty services.

2. TYPES OF WARRANTY POLICIES

2.1. Warranty Policies Based on Warranty Period

Warranty policies have been categorized in various ways based on their characteristics. An example of such a classification was provided by Huang and Zhou [5]. In this classification, two variables distinguish different types of warranty policies: the warranty period and the manner of providing the warranty. The warranty period refers to the length of time during which the product is covered under the warranty. Based on this, there are two types of policies: fixed-period warranties and renewable warranties.

- In fixed-period warranties, the warranty period ends after a specified time, and the product is no longer covered by the warranty.
- In renewable warranties, whenever a product fails during the warranty period, the warranty period is renewed, and a new period replaces the previous one. Thus, for renewable policies, the warranty period restarts with each product failure. In contrast, for fixed-period policies, the product remains covered for the remaining duration of the warranty period after a failure occurs.

To control warranty costs, most manufacturers offer their products with fixed-period (non-renewable) warranties.

2.2. Warranty Policies Based on Cost or Service Provision

A- Free Replacement Warranty (FRW)

In free warranty, customer service is provided free of charge during the warranty period. Under the Free Replacement Warranty (FRW) policy, if a product fails during the warranty period, it is replaced with another product of the same type at no cost, or the manufacturer and the buyer agree on a full refund of the product's price. In both cases, the buyer receives full compensation for the product's failure during the warranty period [2].

B- Cost Sharing Policy (Pro-rata)

Under the Cost Sharing Policy (Pro-rata Warranty), during the warranty period, the manufacturer repairs or replaces the product by charging the customer a portion of the cost. In this policy, the customer bears a specific proportion of the warranty cost. More precisely, under the PRW (Pro-rata Warranty) policy, the manufacturer compensates the buyer for product failure during the warranty period, where the compensation amount is a linear or nonlinear function of the remaining warranty period [2].

C- Hybrid Warranty Policy

A hybrid warranty is formed by combining the Free Replacement Warranty (FRW) and the Cost Sharing Warranty (PRW) [5]. In this hybrid policy, a combination of the FRW policy is applied during the initial period $[0,w_1)$ and the PRW policy is applied during the subsequent period $[w_1,w_2)$, where $w1$ and $w2$ are positive values and $w_1 < w_2$. This hybrid policy means that if a product with a lifetime t fails within the interval $[0,w_1)$, the seller is obligated to provide full compensation. If the product fails within the interval $[w_1, w_2)$, the seller pays a percentage of the product's price as compensation. Finally, if the product's failure occurs after w_2, no compensation is provided. Figure 1 illustrates the diagram of this policy [2].

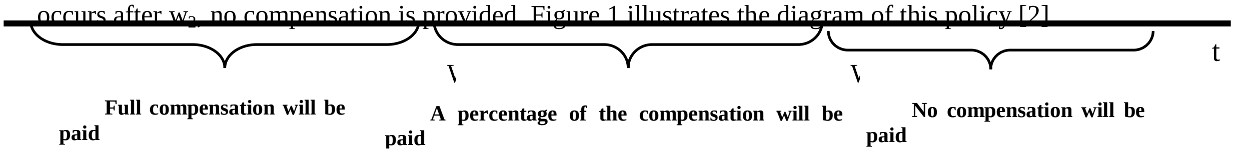

Fig 1. *Diagram of the Hybrid Policy (Mohammadpour and Razavi, 2017)*

In combined warranty policies, the selection of points $w1$ and $w2$ is crucial for determining the boundaries of the two warranties and maximizing the profitability of manufacturers.
It is important to note that the FRW (Free Replacement Warranty) and PRW (Pro-Rata Warranty) policies are special cases of the combined FRW/PRW policy. Specifically, if $w_1 = w_2$, the FRW policy is obtained, and if $w_1=0$, the PRW policy is obtained.

Assuming that SS represents the selling price of a product, the failure cost of this product at the time of failure tt for the manufacturer under the combined FRW/PRW policy is as follows:

$$C_W(t) = \begin{cases} S & 0 \leq t \leq w_1 \\ S\left(\frac{w_2 - t}{w_2 - w_1}\right) & w_1 \leq t < w_2 \\ 0 & t \geq w_2 \end{cases} \quad (1)$$

As can be observed, if the failure time falls within the interval $[0,w_1)$, the full cost is covered, and if the failure time falls within the interval $[w_1, w_2)$, a portion of the product's price is compensated (this function can also take other forms).

2.3. Warranty Policies Based on Correction Method

Another classification that has been examined in warranty policies is related to the research by Eskandar and colleagues[1][6]. This research categorizes warranties based on the method of remediation. The method of remediation refers to the type of service mentioned in the warranty contract that must be performed to fix the product's defect. The manufacturer provides two types of services to the consumer to address the product's defect. These services include product replacement and product repair. Depending on whether the product is repairable or not, a cost-effective solution is chosen [6].

2.4. Warranty Policies Based on Warranty Dimension

A one-dimensional policy is based on either the time and age of the product or its usage and operational level. A two-dimensional policy, on the other hand, incorporates both variables of time and usage. For example, in the case of automobiles, a warranty may only have a time limitation (e.g., two years), which is called a one-dimensional warranty; whereas a two-dimensional warranty includes both time and usage limitations (e.g., two years or 30,000 kilometers of usage).
By studying the mentioned models and integrating existing theories, a relatively comprehensive classification of warranty policies has been proposed, as shown in Figure 2. This classification is presented below.

[1] Iskandar, B.P. and Murthy, D.N.P. and Jack, N

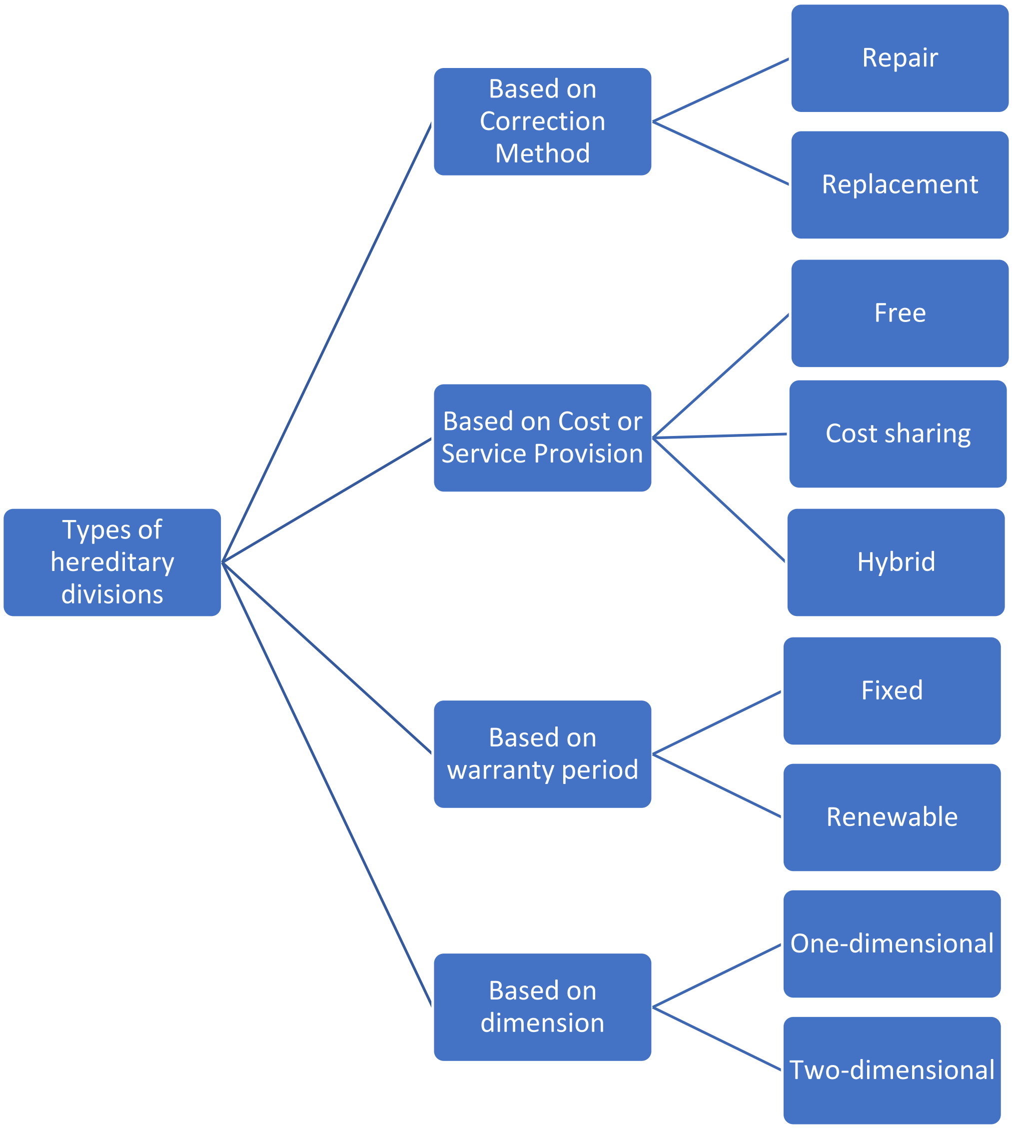

Fig 2. *Classification of Warranty Policies [7].*

By combining each of the above cases, a warranty policy is formed. Since the total number of possible permutations for the above classification is 2×2×3×2=24, 24 different warranty policies can be proposed based on this classification.

3. LITERATURE REVIEW

Although extensive research has been conducted in the field of warranty cost estimation, for the sake of brevity, only a few studies that are more relevant to the current issue and more recent in terms of timing are mentioned here.

Afshari and colleagues [8] in their paper proposed a hybrid approach for joint optimization and extended warranty decisions, considering out-of-warranty product production. They developed an optimization model to maximize the manufacturer's profit.

MoghimiHadji [9] in his research derived the optimal length of initial failure testing periods and warranty periods, considering different types of repairs. This study analyzed the total costs incurred during the initial failure testing and warranty periods from the manufacturer's perspective. The research considered various repair services for a product with a bathtub-shaped failure curve. During the initial testing period, the product experiences minor failures with probability p_1, undergoes minor repairs, and the testing continues. With probability $1-p_1$, the product experiences a major failure, prompting the manufacturer to replace the defective part and restart the initial testing. During the warranty period, the product experiences a major failure with probability $1-p_2$, requiring the manufacturer to perform major/overhaul repairs. The repair and replacement times are considered negligible. The output of this model is the optimal length of the initial performance testing period and the warranty period.

Hooti and his colleagues [10] in their recent research, aimed to find the optimal length of an extended warranty period, considering a limited number of repairs during the warranty period. They proposed a plan for the warranty period length for a repairable system consisting of several minor repairs and service durations. They considered both exponential and Weibull distributions for the product's lifetime. In this model, when the optimal warranty period length and the optimal number of services are determined, the manufacturer's profit is maximized.

Bayan and his team of researchers [11] in their research, compared a new proposed trade-in service policy with the traditional free replacement or repair policy in extended warranties. They concluded that the cost of the new policy would never exceed that of the traditional policy.

Park et al. [12] in their research, aimed to determine an optimal preventive maintenance cycle after the expiration of the warranty and, consequently, provide an optimal post-warranty strategy to the user. In this paper, the impact of several parameters on the optimal strategy was also numerically investigated, and for this purpose, they presented their proposed model.

4. SYMBOLS AND ABBREVIATIONS USED

$F(t)$ Cumulative distribution function of the component's lifetime (probability of failure by time tt)

$f(t)$ Probability density function of the component's lifetime

n Number of system components (number of parts installed in parallel in the system)

$N(t)$ Number of failures over a time period of length tt

W Length of the warranty period (decision variable)

$C(W)$ Warranty cost

$\bar{F}(t)$ Reliability function (probability of failure-free operation until time tt)

c_1 Cost of raw materials/purchasing a component

c_2 Installation and setup cost of a component

c_3 Operational cost per unit time for each component

c_4 Cost of performing minor repairs at the manufacturer's site

p_1 Probability of minor failure during the initial performance testing

$1-p_1$ Probability of major failure during the initial performance testing period

b Length of the initial performance testing period

$h(t)$ Failure rate function, defined as: $h(t)=f(t)/(1-F(t))$

c_5 Cost of system replacement during the initial performance testing period

$N(b)-1$ Number of system replacements until a system successfully completes the period b for the first time

c_6	Cost of minor system repairs at the customer's site during the period [b,b+W]
c_7	Cost of major system repairs at the customer's site during the period [b,b+W]
δ	Degree of repair (ranging between 0 and 1). Here, δ=0 (replacement) and δ=1 (minor repair) is not considered.
q(x)	Linear cost-sharing (pro rata) function during the warranty period

5. MATHEMATICAL MODEL

Consider a repairable system consisting of *n* identical components (with the same lifetime distribution function) arranged in parallel to enhance system reliability. To prevent early system failures at the customer's site, the system undergoes an initial performance test[2] for a duration of *b* time units (the optimal value of which will be calculated later). During this period, the system may experience a minor failure with probability p_1, which is repaired through minor maintenance[3], or a major failure with probability $1 - p_1$, in which case the system must be replaced[4]. Based on these assumptions, the cost function during the initial performance testing period can be expressed as:

$$TC_b = p_1(nc_1 + nc_2 + N(b) \times c_4 + nc_3 b] + (1-p_1)[n(c_1+c_2)N(b) + nc_3\left(\sum_{i=1}^{N(b)-1} x_i + b\right) + c_5(N(b)-1)] \quad (2)$$

where x_i represents the system's lifetime at the i^{th} failure occurring during the initial performance testing period (from the start of the system's operation until time *b*). It can be observed that the cost function consists of two parts:

Part 1: If the system experiences a minor failure, the cost function is given by the first term in Equation (2). This term includes four components:

- The cost of minor repair is c_4, and the number of replacements during the initial testing period is *N(b)*. Thus, the cost for minor repairs during this period is $N(b) \times c_4$
- Additionally, there are costs for purchasing raw materials and installation for each component, which amount to $n(c_1 + c_2)$
- The operational cost per unit time for minor repairs for each component is c_3. Since the initial testing period is *b*, the total operational cost for the system in this scenario is nc_3b.

Therefore, if the system experiences a minor failure during the initial performance testing period and is repaired through minor maintenance, the cost to the manufacturer will be:

$$nc_1 + nc_2 + N(b) \times c_4 + nc_3 b \quad (3)$$

Part 2: If the number of failures is *N(b)*, and the system is replaced upon encountering a major failure, the installation cost for new systems will be $n(c_1 + c_2)\ N(b)$. The replacement cost is c_5, and since there are *N(b)-1* replacements, the total replacement cost will be $c_5\ (N(b) - 1)$. The operational cost for failed systems will be $nc_3\left(\sum_{i=1}^{N(b)-1} x_i\right)$ and the operational cost for the last system that completes the period without failure is $nc_3\, b$.

Thus, if the system fails during the warranty period at the customer's site, the warranty service costs for the manufacturer during this period will be:

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[2] Burn-in Test
[3] Minimal Repair
[4] Replacement

$$n(c_1 + c_2)N(b) + nc_3\left(\sum_{i=1}^{N(b)-1} x_i + b\right) + c_5(N(b) - 1) \tag{4}$$

Note: If the components are identical (with the same lifetime distribution function) and the reliability of each component is *r*, the probability of failure for each component is:

$$R' = 1 - r \tag{5}$$

Assuming the components of this parallel system are identical and independent, the system's failure probability will be:

$$R'_s = (1 - r). \ldots . . (1 - r) = (1 - r)^n \tag{6}$$

Therefore, the system's reliability (probability of functioning) will be:

$$R_s = 1 - (1 - r)^n \tag{7}$$

5.1 Average System Cost During the Period (0, b)

Since the system's failure rate function is given by:

$$h_s(t) = \frac{f_s(t)}{\bar{F_s}(t)} \tag{8}$$

the number of system failures during the initial performance testing period can be calculated as:

$$N_s(b) = \int_0^b h_s(t)\, dt \tag{9}$$

We know that $N_s(b)$ follows a geometric distribution because:

$$P(N_s(b) = k) = F_s^{k-1}(b).\bar{F}_s(b) \tag{10}$$

Thus, the expected value of $N_s(b)$ can be calculated from the geometric distribution as:

$$E(N_s(b)) = \frac{1}{\bar{F_s}(b)} \tag{11}$$

Based on the well-known Wald's equation[5], we have:

$$E\left(\sum_{i=1}^{N(b)-1} x_i\right) = E\big(N_s(b)\big) * E(x_i) - E\big(x_{N_s(b)}\big) = \frac{1}{\bar{F_s}(b)} * \int_0^b \bar{F}_s(b)\, dt \ - b \tag{12}$$

Therefore, Equation (2) can be rewritten as:

$$E(Tc_b) = p_1\left(n(c_1 + c_2) + c_4\frac{1}{\bar{F}_s(b)} + nc_3 b\right) + (1 - p_1)\left[n(c_1 + c_2)\frac{1}{\bar{F}_s(b)} + nc_3\left(\frac{\int_0^b \bar{F}_s(t)dt}{\bar{F}_s(b)}\right) + c_5\frac{F_s(b)}{\bar{F}_s(b)}\right] \tag{13}$$

Using the above equation, the average cost for the manufacturer during the burn-in test period (0, b] can be calculated.

5.2 Average System Cost During the Warranty Period [b, b + w]

If the system successfully passes the initial performance test, it enters the market with a proposed warranty period of length *w*. During this period, if the system experiences a minor failure with probability p_2, the manufacturer performs minor repairs. If it experiences a major failure with probability $1 - p_2$, the manufacturer performs a generl repair. Note that, based on the manufacturer's warranty service definition,

[5] Wald's identity

only *p* percent of warranty claims/failures are covered, and not all failures during this period are covered. For example, failures caused by physical damage or electrical failures due to liquid spills are not covered.

Various forms of linear cost-sharing (pro rata) functions can be found in the literature. In this model, the following form is arbitrarily used, although other models would not affect the problem's formulation here:

$$q(x) = 1 - \frac{x}{b+w} \qquad b < x < b + w \tag{14}$$

The cost function during the warranty period is then given by:

$$TC_W = p\left\{p_2\left[c_6\left(1-\frac{x}{b+w}\right)\int_b^{b+w} h_s(t)dt\right)\right] + (1-p_2)\left[c_7(1-\delta)\left(1-\frac{x}{b+w}\right)\left(N_g(w)\right)\right]\right\} \tag{15}$$

where $N_g(w)$ is the expected number of failures under the condition of general repair. The above cost function assumes that failures are covered under warranty. The equation shows that the cost function consists of two parts:

Part 1: If the system experiences a minor failure, the cost function is given by the first term in Equation (15). For minor failures, the repair cost during the warranty period is c_6, and a linear cost-sharing function is used. Given the system's hazard rate function during this period, the number of components undergoing minor repairs is $N_s(w) = \int_b^{b+w} h_s(t)\,dt$ Thus, the total cost in this case is the product of the total warranty cost under linear cost-sharing and the number of components undergoing minor repairs.

Part 2: If the system experiences a major failure and requires general repair, the repair degree is $(1 - \delta)$, and the on-site repair cost during the warranty period is c_7. A linear pro rata function is used, and given the system's hazard rate function during this period, the number of components undergoing general repairs is $N_g(w)$. Thus, the total cost in this case is the product of the total warranty cost under linear pro rata and the number of components undergoing general repairs. Note that when the system undergoes general repair, its lifetime decreases proportionally to the repair degree, so the number of failures cannot be simply calculated by integrating over the desired interval.

The expected number of failures when the system undergoes general repair (or general repair) during the warranty period is denoted by $N_g(w)$, and its average value can be calculated from the following equation [13]:

$$E\left(N_g(w)\right) = M_g(w) = Q(t|0) + \int_0^t Q(t-x|x)m(x)dt \tag{16}$$

where:

$$Q(x) = \int_0^t \frac{f(x+y)}{\bar{F}(x)}dy \tag{17}$$

Therefore, the objective function for the average cost over the entire burn-in and warranty periods can be calculated as:

$$\min(ATC) = \frac{TC_b + TC_w}{W} \tag{18}$$

Note that, due to the small length of period b compared to the warranty period, it is neglected in the calculation of the total average cost. The goal here is to calculate the values of b and w by minimizing the total average cost. Constraints can also be applied in this model, such as available volume constraints. If the volume of each component is v_i and the total available volume for the system is V, then:

$$\sum_{i=1}^{n} v_i \le V \tag{19}$$

Another constraint is the minimum desired system reliability. If the reliability of each component is r and the system's overall reliability is R_s, then:

$$R_s \ge 1 - (1-r)^n \tag{20}$$

6. NUMERICAL EXAMPLE AND ANALYSIS

To demonstrate the capabilities of the proposed model, we consider a system consisting of one or more independent components with the following lifetime density function:

$$f(t) = t\left(1 + e^{t^2}\right).e^{0.5\left(1-t^2-e^{t^2}\right)} \tag{21}$$

The general form of the failure rate function is given by:

$$h(t) = kcL\, t^{c-1} + (1-k)b\, t^{b-1}B.e^{Btb} \tag{22}$$

In this example, the values of L and B are set to 1, k = 0.5, and c = b = 2. This five-parameter failure rate function was first introduced by Dhillon in 1979. When plotted, this failure rate function can produce various shapes, including increasing failure rates or bathtub-shaped failure rates, depending on the parameter values. Here, *L* and *B* are scale parameters, *c* and *b* are shape parameters, and *k* is a constant between 0 and 1. The failure rate function with the above parameters is shown in the following figure:

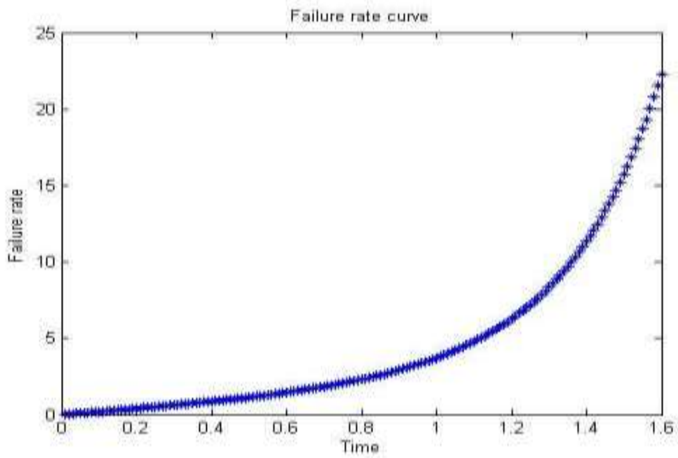

Fig 3. *Failure rate function (B=1, L=1, k=0.5, c=2, b=2)*

This graph shows a failure rate that starts with a gentle slope and then increases rapidly over time. The probability of a minor failure during the initial performance testing period (burn-in period) is assumed to be 0.9, the probability of a minor failure during the warranty period is 0.7, and the probability that a failure is covered under warranty is 0.8 (these values can be adjusted as needed). The cost coefficients for the system are defined as follows:

- c_1: Cost of raw materials/purchasing a component = 50 monetary units
- c_2: Installation/setup cost for a component = 3 monetary units
- c_3: Operational cost per unit time = 1 monetary unit
- c_4: Cost of minor repair for a component at the manufacturer's site = 5 monetary units
- c_5: Cost of replacing a component during the initial performance testing period = 53 monetary units (includes raw material and installation costs)
- c_6: Cost of minor repair for a component at the customer's site during the warranty period = 5 monetary units
- c_7: Cost of general repair at the customer's site during the warranty period = 10 monetary units

One of the constraints of the problem is the minimum system reliability, which is set to 99% in this example. If the designed system's reliability falls below this threshold (99%), the system is not acceptable. Additionally, the maximum available space for installing the system is 130 volume units (system volume constraint), and the volume of each component is assumed to be 50 units.

For the numerical solution of this model, the desired range for the burn-in test period is assumed to be between 5% and 30% of the component's average lifetime, and the warranty period length is assumed to be between 10% and 100% of the component's average lifetime (only the average total costs for initial performance testing with 0.05 and 0.1 of the product's average lifetime are shown in the table below). The repair degree is arbitrarily set to 0.3, 0.5, and 0.7, where δ=0 means replacing a defective component, and δ=1 means performing minor repair on the component. Any other values can be chosen for the defined parameters, and the problem can be solved accordingly.

Using the above-defined values, the problem was solved using MATLAB programming, and the results are presented in the following tables:

Table 1. *Average total cost for the defined problem*
# of parts

Average TC	# of parts	Delta	w	b	Average TC	# of parts	Delta	w	b
578.19	1	0.3	0.1	0.1	576.40	1	0.3	0.1	0.05
577.95	1	0.5	0.1	0.1	576.21	1	0.5	0.1	0.05
577.70	1	0.7	0.1	0.1	576.01	1	0.7	0.1	0.05
1107.12	2	0.3	0.1	0.1	1106.05	2	0.3	0.1	0.05
1107.09	2	0.5	0.1	0.1	1106.04	2	0.5	0.1	0.05
1107.06	2	0.7	0.1	0.1	1106.02	2	0.7	0.1	0.05
289.87	1	0.3	0.2	0.1	288.90	1	0.3	0.2	0.05
289.46	1	0.5	0.2	0.1	288.55	1	0.5	0.2	0.05
289.05	1	0.7	0.2	0.1	288.20	1	0.7	0.2	0.05
553.90	2	0.3	0.2	0.1	553.28	2	0.3	0.2	0.05
553.79	2	0.5	0.2	0.1	553.20	2	0.5	0.2	0.05
553.67	2	0.7	0.2	0.1	553.11	2	0.7	0.2	0.05
232.35	1	0.3	0.3	0.1	231.55	1	0.3	0.3	0.05
231.86	1	0.5	0.3	0.1	231.12	1	0.5	0.3	0.05
231.37	1	0.7	0.3	0.1	230.69	1	0.7	0.3	0.05
443.45	2	0.3	0.3	0.1	442.88	2	0.3	0.3	0.05
443.26	2	0.5	0.3	0.1	442.74	2	0.5	0.3	0.05
443.06	2	0.7	0.3	0.1	442.60	2	0.7	0.3	0.05
146.47	1	0.3	0.4	0.1	145.91	1	0.3	0.4	0.05
145.72	1	0.5	0.4	0.1	145.22	1	0.5	0.4	0.05
144.97	1	0.7	0.4	0.1	144.54	1	0.7	0.4	0.05
278.44	2	0.3	0.4	0.1	277.95	2	0.3	0.4	0.05

277.94	2	0.5	0.4	0.1	277.52	2	0.5	0.4	0.05
277.43	2	0.7	0.4	0.1	277.09	2	0.7	0.4	0.05
118.04	1	0.3	0.5	0.1	117.57	1	0.3	0.5	0.05
117.15	1	0.5	0.5	0.1	116.74	1	0.5	0.5	0.05
116.24	1	0.7	0.5	0.1	115.89	1	0.7	0.5	0.05
223.83	2	0.3	0.5	0.1	223.39	2	0.3	0.5	0.05
223.10	2	0.5	0.5	0.1	222.74	2	0.5	0.5	0.05
222.37	2	0.7	0.5	0.1	222.09	2	0.7	0.5	0.05
107.72	1	0.3	0.6	0.1	107.29	1	0.3	0.6	0.05
106.77	1	0.5	0.6	0.1	106.40	1	0.5	0.6	0.05
105.80	1	0.7	0.6	0.1	105.48	1	0.7	0.6	0.05
203.97	2	0.3	0.6	0.1	203.60	2	0.3	0.6	0.05
203.16	2	0.5	0.6	0.1	202.85	2	0.5	0.6	0.05
202.35	2	0.7	0.6	0.1	202.10	2	0.7	0.6	0.05
85.64	1	0.3	0.7	0.1	85.26	1	0.3	0.7	0.05
84.65	1	0.5	0.7	0.1	84.29	1	0.5	0.7	0.05
83.60	1	0.7	0.7	0.1	83.27	1	0.7	0.7	0.05
161.10	2	0.3	0.7	0.1	160.89	2	0.3	0.7	0.05
160.22	2	0.5	0.7	0.1	160.04	2	0.5	0.7	0.05
159.34	2	0.7	0.7	0.1	159.17	2	0.7	0.7	0.05
75.97	1	0.3	0.8	0.1	75.38	1	0.3	0.8	0.05
75.25	1	0.5	0.8	0.1	74.53	1	0.5	0.8	0.05
75.14	1	0.7	0.8	0.1	73.81	1	0.7	0.8	0.05
141.32	2	0.3	0.8	0.1	141.11	2	0.3	0.8	0.05
140.53	2	0.5	0.8	0.1	140.31	2	0.5	0.8	0.05
139.78	2	0.7	0.8	0.1	139.51	2	0.7	0.8	0.05
70.63	1	0.3	0.9	0.1	**68.84**	1	0.3	0.9	0.05
76.36	1	0.5	0.9	0.1	69.85	1	0.5	0.9	0.05
923.02	1	0.7	0.9	0.1	116.51	1	0.7	0.9	0.05
126.27	2	0.3	0.9	0.1	125.88	2	0.3	0.9	0.05

126.06	2	0.5	0.9	0.1	125.33	2	0.5	0.9	0.05
143.84	2	0.7	0.9	0.1	126.75	2	0.7	0.9	0.05
93.28	1	0.3	1	0.1	71.64	1	0.3	1	0.05
11361.02	1	0.5	1	0.1	402.35	1	0.5	1	0.05
1813565161244	1	0.7	1	0.1	281547341	1	0.7	1	0.05
116.37	2	0.3	1	0.1	114.62	2	0.3	1	0.05
217.94	2	0.5	1	0.1	122.40	2	0.5	1	0.05
1579520190	2	0.7	1	0.1	616723	2	0.7	1	0.05

As can be inferred from the data in the table, the minimum average total cost is achieved for the shortest initial performance testing period (equivalent to 0.05 of the product's average lifetime), a relatively long warranty period (90% of the product's average lifetime), one component (since its reliability exceeds 99%), and a repair degree δ=0.3.

Since the initial performance testing process is costly, the model aims to minimize the length of this period as much as possible. Given the constraint of the minimum desired reliability in this example, the system achieves the required reliability with one component, so the model does not insist on increasing the number of components to two. This is because increasing the number of components from one to two can raise the cost from 68.82 monetary units to 125.88 monetary units.

The choice of a low value for δ (0.3) is because performing a good general repair can significantly reduce the failure rate in the future, and a lower number of failures will reduce repair costs over the component's lifetime. For example, if a poor general repair is performed on the system in this example, the average total cost increases from 68.84 monetary units to 116.51 monetary units.

From a brief examination of the table, it can be observed that, keeping other parameters constant, as the length of the initial performance testing period increases, the average total cost gradually increases. For example, consider the following table and the graph that follows (in all cases, the number of components is 2):

Table 2. *Gradual Increase in Average Total Cost with an Increase in the Initial Performance Testing Period*

W	Delta	ATC1	ATC2	ATC3	ATC4	ATC5	ATC6
		b=0.05	b=0.1	b=0.15	b=0.2	b=0.25	b=0.3
0.1	0.3	1106.05	1107.12	1108.3	1109.6	1111.1	1113.1
0.1	0.5	1106.04	1107.09	1108.2	1109.5	1111.0	1112.9
0.1	0.7	1106.02	1107.06	1108.2	1109.4	1110.9	1112.8
0.2	0.3	553.28	553.90	554.6	555.4	556.3	557.4
0.2	0.5	553.20	553.79	554.4	555.1	556.0	557.1
0.2	0.7	553.11	553.67	554.3	554.9	555.7	556.7

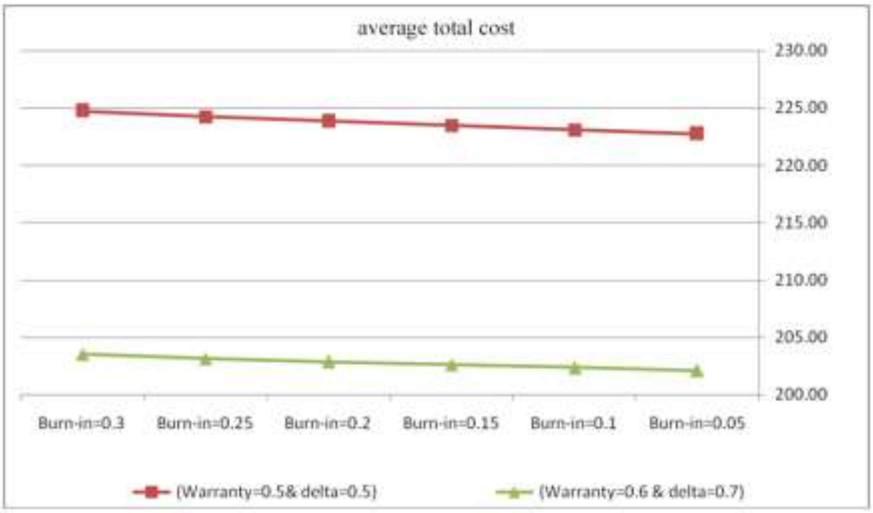

Fig 4. *Graph of Average Total Cost vs. Initial Performance Testing Period (Number of Components: 2)*

By examining the numbers in the table closely, it can be observed that increasing the length of the warranty period (while keeping other parameters constant) leads to a decrease in the average total cost. For example, consider Table 3, which shows the average total cost for different warranty period lengths (with δ=1 and the number of components fixed at 2):

Table 3. *Average Total Cost for Different Warranty Period Lengths (δ=1, Number of Components: 2)*

Warranty	ATC1	ATC2	ATC3	ATC4	ATC5	ATC6
	Burn-in=0.05	Burn-in=0.1	Burn-in=0.15	Burn-in=0.2	Burn-in=0.25	Burn-in=0.3
0.1	1106.05	1107.12	1108.3	1109.6	1111.1	1113.1
0.2	553.28	553.90	554.6	555.4	556.3	557.4
0.3	442.88	443.45	444.1	444.7	445.6	446.5
0.4	277.95	278.44	279.0	279.5	280.1	280.9
0.5	223.39	223.83	224.3	224.7	225.2	225.7
0.6	203.60	203.97	204.3	204.7	205.1	205.6
0.7	160.89	161.10	161.3	161.6	161.9	162.3
0.8	141.11	141.32	141.6	141.9	142.5	143.5
0.9	125.88	126.27	126.9	128.1	131.1	142.3

The graph corresponding to the values in the table above is shown in the figure below. As can be seen in the figure, due to the negligible differences in costs, the graphs are almost overlapping.

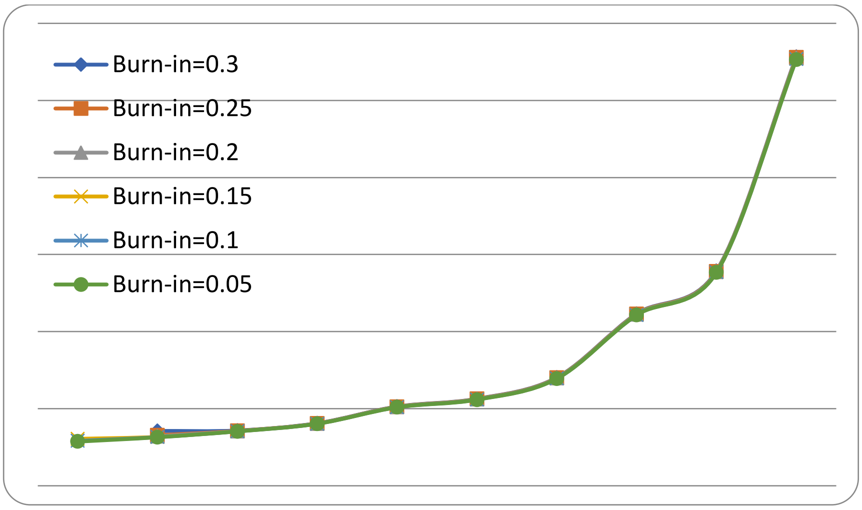

Figure 5. *Average total cost for different warranty period lengths (δ=1, number of components: 2)*

The only exception in this example is the warranty period equal to the full average lifetime (100%). The reason for this is the sudden rise in the failure rate over time, which, during this period, leads to a sharp increase in the number of failures and, consequently, a significant increase in system repair costs.

7. CONCLUSION AND RECOMMENDATIONS

In this model, a system consisting of multiple components arranged in parallel to enhance system reliability is considered. To prevent early failures at the customer's site, the components undergo an initial performance testing period before being released to the market. Components that successfully pass this testing period are then sold. Since all costs during the initial testing period are borne by the manufacturer, this process is expensive. However, to build a good reputation with customers and prevent them from experiencing early random failures, the manufacturer conducts such tests to avoid delivering defective products.

Generally, a short initial testing period is effective in preventing many early failures at the customer's site. If the length of this period increases, the costs will also increase. On the other hand, the warranty period should be close to the average lifetime of the product. If this period is too long, the rapid increase in the failure rate will lead to a sharp rise in costs.

The choice of an appropriate repair degree also has a significant impact on costs during the warranty period. If the system undergoes proper general repairs after a failure, future early failures can be prevented, reducing the average repair costs over time. If a minor repair policy is adopted, subsequent failures are likely to occur in a short time, increasing the average repair costs and reducing system availability, which may lead to customer dissatisfaction.

In this research, a linear cost-sharing model was used for the warranty period. Future studies could explore non-linear cost-sharing models or other models. Additional constraints could also be added to the model. In this study, the components were arranged in parallel. Future work could also consider series configurations. This research focused on a single-stage warranty. Future studies could investigate multi-stage warranties (e.g., two or more stages).

One of the main limitations of this research is the exclusion of series systems and systems with a large number of components, as the calculations related to the cost function and other parameters become complex and time-consuming.

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