Weak character amenability of Banach algebras

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ABSTRACT

In the present paper we first investigate the relations between weak character amenability of A^{**} and weak character amenability of A, where A^{**} is the second dual algebra of A endowed with the first or second Arens product. Then we introduce and study the concept of weak (φ,φ)-amenability of a Banach algebra A, where $\varphi \in \Delta(A)$, the character space of A. We give some interesting examples and establish a characterization of weak (φ,φ)-amenability. Moreover we prove some hereditary properties and investigate weak (φ,φ)-amenability for projective tensor product $A \widehat{\otimes} B$.

Keywords: Derivation, weak φ-amenability, weak (φ,φ)-amenability, weak Δ-amenability, Banach algebra.

1. *INTRODUCTION*

The notion of weak amenability was first introduced by Bade, Curtis and Dales in [1] for commutative Banach algebras, and was extended to the noncommutative case by Johnson in [7] (see also, [3], [4], [8] and [13]). Gronbaek in [3], investigated properties of weakly amenable Banach algebras, in particular for two commutative weakly amenable Banach algebras A and B he showed that the Banach algebra $A \widehat{\otimes} B$ is weakly amenable. Suppose that A is a Banach algebra and X is a Banach A -bimodule. A derivation from A into X is a linear operator $D: A \longrightarrow X$ satisfying

$$D(ab) = D(a)\cdot b + a\cdot D(b) \qquad (a,b \in A).$$

A derivation D is inner if there is $x_0 \in X$ such that $D(a) = a\cdot x_0 - x_0 \cdot a$ for $a \in A$. A Banach algebra A is called weakly amenable if every continuous derivation $D: A \longrightarrow A^*$ is inner.

Kaniuth, Lau and Pym in [9], investigated the concept of φ-amenability for an arbitrary Banach algebra A, where $\varphi \in \Delta(A)$, generalizing left amenability for Lau algebras of [12]. A is called (left) φ-amenable if there exists $m \in A^{**}$ such that $m(\varphi) = 1$ and $m(f\cdot a) = \varphi(a)f$ for all $a \in A$ and $f \in A^*$. They also gave several characterizations of φ-amenability. For instance, they have shown that A is called φ-amenable if for every Banach A-module X with the left module action $a\cdot x = \varphi(a)x$ $(a \in A, x \in X)$ (and no restriction on the right module action), every continuous derivation from A into X^* is inner.

Moreover, the notion of (right) character amenability was introduced and studied by Monfared in [16]. He called a Banach algebra A character amenable if it is both left and right character amenable.

Recently, in [18], the authors have introduced and investigated the concept of weak character amenability of a Banach algebra A as a new notion of amenability. They showed that if A has a bounded left approximate identity, then A is weakly character amenable but the converse is not true. For more information about φ-amenability and related concepts we refer readers to [19, 20, 21, 22].

This paper is organized as follows:

In section 2 for a Banach algebra A we investigate the relations between weak character amenability of A^{**} and weak character amenability of A.

In section 3 we characterize weak (φ,φ)-amenability of Banach algebras and compare the notion of weak (φ,φ)-amenability with some different versions of amenability. We show that the class of weakly

(φ,φ)-amenable Banach algebras is larger than that of φ -amenable Banach algebras (resp. weakly amenable commutative Banach algebras). Moreover we give some illuminating examples and prove some hereditary properties.
In section 4, for Banach algebras A and B, we investigate relation between weak $(\varphi\otimes\psi,\varphi\otimes\psi)$-amenability of $A\widehat{\otimes}B$ and weak φ-amenability of A and weak ψ-amenability of B, where $\varphi\in\Delta(A)$ and $\psi\in\Delta(B)$.

2. Weak φ-amenability

We commence this section with the following definition:
Let A be a Banach algebra and let A_φ be a Banach A-bimodule with the actions defined by

$$a\cdot x=\varphi(a)x,\quad x\cdot a=xa\quad(a,x\in A).$$

Moreover, $(A_\varphi)^*$ is a Banach A-bimodule with the left and right actions defined by

$$a\cdot f=af,\quad f\cdot a=\varphi(a)f\quad(a\in A,f\in A^*).$$

Here $af\in A^*$ defined by $(af)(b)=f(ba)$ for all $b\in A$ (see [18]).

Definition 2.1. *[18] A Banach algebra A is called weakly φ-amenable if every derivation $D:A\longrightarrow(A_\varphi)^*$ is inner. Also A is called weakly character amenable whenever A is weakly φ-amenable for all $\varphi\in\Delta(A)\cup\{0\}$.*

For a Banach algebra A, $A^{**}$is a Banach algebra under two Arens products.
Proposition 2.2. *Suppose that A_1 and A_2 are Banach algebras and $T_1:A_1\longrightarrow A_2$ and $T_2:A_2\longrightarrow A_1$ are continuous homomorphisms such that $T_1\circ T_2=I_{A_2}$, then the following statements are valid:*
(i) *If A_1 is weakly character amenable, then so is A_2;*
(ii) *If ${A_1}^{**}$ is weakly character amenable, then so is ${A_2}^{**}$.*

Proof. (i) *Suppose that A is weakly character amenable and $\psi\in\Delta(A_2)\cup\{0\}$. Let $D:A_2\longrightarrow(A_{2_\varphi})^*$ be a continuous derivation. Define $\widetilde{D}:A_1\longrightarrow(A_{1_{\psi\circ T_1}})^*$ by $\widetilde{D}(a_1)={T_1}^*\circ D\circ T_1(a_1)$ $(a_1\in A_1)$. It is easy to check that $\widetilde{D}$ is a continuous derivation. Therefore there exists $f\in(A_{1_{\psi\circ T_1}})^*$ such that*

$$\widetilde{D}(a_1)=a_1\cdot f-\psi\circ T_1(a_1)f\quad(a_1\in A_1).$$

Since ${T_2}^*\circ{T_1}^*={I_{B^*}}$, for every $a_2\in A_2$, we have

$$D(a_2)={T_2}^*\circ{T_1}^*\circ D\circ T_1\circ T_2(a_2)={T_2}^*(\widetilde{D}(T_2(a_2)))={T_2}^*(T_2(a_2)\cdot f-\psi\circ T_1(T_2(a_2))f)$$
$$=a_2\cdot{T_2}^*(f)-\psi(a_2){T_2}^*(f).$$

This means that D is inner and so A_2 is weakly ψ-amenable. Therefore A_2 is weakly character amenable.
(ii) *Since ${T_1}^{**}\circ{T_2}^{**}={I_{B^{**}}}$, the proof is similar to* (i). □

Corollary 2.3. *Let A be a Banach algebra such that $A=B\oplus I$ for a closed subalgebra B and closed ideal I. Suppose that A is weakly character amenable, then so is B.*
Proof. *Suppose that A is weakly character amenable. Let $T_1:A\longrightarrow B$ be the natural projection and $T_2:B\longrightarrow A$ be define by $T_2(b)=(b,0)$ $(b\in B)$. Since T_1 and T_1 are continuous homomorphisms, from Proposition 2.2, it follows that B is weakly character amenable.* □

A Banach algebra A is said to be dual if there is a closed submodule A_* of A^* such that $A=(A_*)^*$.
Theorem 2.4. *Let A be a dual Banach algebra. If A^{**} is weakly character amenable, then so is A.*
Proof. *Let $\varphi\in\Delta(A)\cup\{0\}$ and $D:A\longrightarrow(A_\varphi)^*$ be a continuous derivation. Let $i:A_*\longrightarrow A^*$ be the canonical embedding and i^* be the adjoint of i. Obviously i is norm-continuous, hence i^* is $w*$-continuous. Also one can easily show that*

$$i^*(a^{**}b^{**})=i^*(a^{**})i^*(b^{**})\quad(a^{**},b^{**}\in A^{**}).\qquad(1)$$

Let $\widetilde{D}:A^{**}\longrightarrow({A^{**}}_{\varphi\circ i^*})^*$ defined by $\widetilde{D}(a^{**})=i^{**}\circ D\circ i^*(a^{**})$ $(a^{**}\in A^{**})$. By (1) *for every $a^{**},b^{**},c^{**}\in A^{**}$, we have*

$$\langle c^{**}, \widetilde{D}(a^{**}b^{**})\rangle = \langle c^{**}, i^{**}\circ D\circ i^{*}(a^{**}b^{**})\rangle = \langle i^{*}(c^{**}), D(i^{*}(a^{**})i^{*}(b^{**}))\rangle$$
$$= \langle i^{*}(c^{**}), \varphi(i^{*}(b^{**}))D(i^{*}(a^{**})) + i^{*}(a^{**})\cdot D(i^{*}(b^{**}))\rangle$$
$$= \langle c^{**}, \varphi(i^{*}(b^{**}))i^{**}\circ D\circ i^{*}(a^{**}) + i^{**}(i^{*}(a^{**})\cdot D(i^{*}(b^{**})))\rangle$$
$$= \langle c^{**}, \varphi\circ i^{*}(b^{**})\widetilde{D}(a^{**}) + a^{**}\cdot \widetilde{D}(b^{**})\rangle.$$

Thus $\widetilde{D}$ is a continuous derivation. Now weak character amenability of A^{**}, implies that there exists $f\in (A^{**}{}_{\varphi\circ i^{*}})^{*}$ such that

$$\widetilde{D}(a^{**}) = a^{**}\cdot f - \varphi\circ i^{*}(a^{**})f \quad (a^{**}\in A^{**}).$$

The canonical mapping $\pi: A\longrightarrow A^{**}$ is an A-bimodule morphism, and hence is $\pi^{*}: A^{***}\longrightarrow A^{*}$. Set $g=\pi^{*}(f)$. Then for every $a_1, a_2\in A$, we have

$$\langle a_2, D(a_1)\rangle = \langle i^{*}(\widehat{a_2}), D(i^{*}(\widehat{a_1}))\rangle = \langle \widehat{a_2}, i^{**}\circ D\circ i^{*}(\widehat{a_1})\rangle = \langle \widehat{a_2}, \widehat{a_1}\cdot g - \varphi\circ i^{*}(\widehat{a_1})g\rangle$$
$$= \langle \pi(a_2), \widehat{a_1}\cdot g - \varphi\circ i^{*}(\widehat{a_1})g\rangle = \langle a_2, \pi^{*}(\widehat{a_1}\cdot g - \varphi\circ i^{*}(\widehat{a_1})g)\rangle = \langle a_2, a_1\cdot g - \varphi(a_1)g\rangle.$$

This means that D is inner. So A is weakly φ-amenable and this completes the proof. □

3. *Weak $(\varphi.\varphi)$-amenability and some hereditary properties*

We start this section with the following:

Let A be a Banach algebra and $\varphi\in\Delta(A)$. If X is an arbitrary Banach space, then X can be viewed as Banach left or right A-module by the following actions:

$$a\cdot x = \varphi(a)x, \quad x\cdot a = \varphi(a)x \quad (a\in A,\ x\in X),$$

which we denote by $X_{(\varphi,\varphi)}$.

Definition 3.1. *Let A be a Banach algebra and $\varphi\in\Delta(A)$. We say that A is weakly (φ,φ)-amenable if every continuous derivation $D: A\longrightarrow (A_{(\varphi,\varphi)})^{*}$ is zero. We also say that A is called weakly Δ-amenable if A is weakly (φ,φ)-amenable for every $\varphi\in\Delta(A)$.*

Let S be a semigroup. A semi-character on S is a non-zero homomorphism $\emptyset: S\longrightarrow\overline{D}$. The space of semi-characters on S is denoted by Φ_S. The semi-character $\emptyset_S: S\longrightarrow\overline{D}$, defined by $\emptyset_S(t)=1$ $(t\in S)$, is called the augmentation character on S. For every $\emptyset\in\Phi_S$, define $\widehat{\emptyset}: l^1(S)\longrightarrow\mathbb{C}$ by

$$\widehat{\emptyset}(\mathrm{f}) = \sum_{s\in S}\emptyset(s)f(s) \quad (\mathrm{f} = \sum_{s\in S}\mathrm{f(s)}\delta_s\in l^1(S)).$$

It is easily verified that $\widehat{\emptyset}\in\Delta(l^1(S))$. and every character on $l^1(S)$ arises in this way. Indeed, $\Delta\big(l^1(S)\big) = \{\widehat{\emptyset}: \emptyset\in\Phi_S\}$.

Example 3.2. *Let $S=\mathbb{Z}^{+}$ be the set of positive integer numbers, with the product*

$$\mathrm{m}\circ\mathrm{n} = \begin{cases}\mathrm{m} & \text{if } \mathrm{m}=\mathrm{n},\\ 0 & \text{if } \mathrm{m}\neq\mathrm{n}.\end{cases}$$

It is easy to see that $\Phi_S=\{\emptyset_t: t\in\mathbb{N}\}\cup\{\emptyset_S\}$ where $\emptyset_S$ is the augmentation character and for each $t\in\mathbb{N}$,

$$\emptyset_t(\mathrm{m}) = \begin{cases}1 & \text{if } \mathrm{m}=\mathrm{t},\\ 0 & \text{if } \mathrm{m}\neq\mathrm{t},\end{cases}$$

for all $m\in S$. Let $\varphi=\widehat{\emptyset_t}$ $(t\in\mathbb{N})$ and $D: l^1(S)\longrightarrow (l^1(S)_{(\varphi,\varphi)})^{*}$ be a continuous derivation. Then

$$D(\delta_{\mathrm{t}}) = \mathrm{D}\big(\delta_{\mathrm{t}}{}^2\big) = \varphi(\delta_{\mathrm{t}})\mathrm{D}(\delta_{\mathrm{t}}) + \varphi(\delta_{\mathrm{t}})\mathrm{D}(\delta_{\mathrm{t}}) = \emptyset_t(t)D(\delta_{\mathrm{t}}) + \emptyset_t(t)D(\delta_{\mathrm{t}}) = 2D(\delta_{\mathrm{t}}).$$

It follows that $D(\delta_{\mathrm{t}})=0$. Also for every $m\neq t$, we have

$$D(\delta_{\mathrm{m}}) = \mathrm{D}\big(\delta_{\mathrm{m}}{}^2\big) = \varphi(\delta_{\mathrm{m}})\mathrm{D}(\delta_{\mathrm{m}}) + \varphi(\delta_{\mathrm{m}})\mathrm{D}(\delta_{\mathrm{m}}) = \emptyset_t(m)D(\delta_{\mathrm{m}}) + \emptyset_t(m)D(\delta_{\mathrm{m}}) = 0.$$

Thus $D(\delta_{\mathrm{m}})=0$ $(m\in S)$. Consequently $D=0$. So $l^1(S)$ is weakly (φ,φ)-amenable. Similarly, for $\varphi=\widehat{\emptyset_S}$ one can show that every continuous derivation $D: l^1(S)\longrightarrow (l^1(S)_{(\varphi,\varphi)})^{*}$ is zero. Therefore $l^1(S)$ is weakly Δ-amenable.

Example 3.3. *Let X be a Banach space and take $\varphi\in X^{*}\backslash\{0\}$ with $\|\varphi\|\leq 1$. Define a product on X by $ab=\varphi(b)a$ $(a,b\in X)$. With this product X is a Banach algebra, which denoted by $A_\varphi(X)$ (see Example 2.4 of [17]). It is clear that $\Delta(A_\varphi(X))=\{\varphi\}$. We show that $A_\varphi(X)$ is weakly (φ,φ)-amenable. To see this, suppose that $D: A_\varphi(X)\longrightarrow (A_\varphi(X)_{(\varphi,\varphi)})^{*}$ is a continuous derivation. Then for every $a,a'\in A_\varphi(X)$, we have*

$$\varphi(a)D(a') = D(a'a) = \varphi(a')D(a) + \varphi(a)D(a').$$

It follows that $\varphi(a')D(a)=0$. Choose $a_0\in A_\varphi(X)$ such that $\varphi(a_0)=1$. Hence for every $a\in A_\varphi(X)$,

$$D(a) = \varphi(a_0)D(a) = 0.$$

Consequently $D = 0$. *Therefore* $A_\varphi(X)$ *is weakly* (φ, φ)*-amenable.*

One can easily show that every φ*-amenable Banach algebra* A *is weakly* (φ, φ)*-amenable. If we take a Banach space* X *with* $\dim(X) > 1$ *in the above example, then by Example 2.4 of [17], we conclude that* $A_\varphi(X)$ *is not* φ*-amenable but is weakly* (φ, φ)*-amenable. So the class of weakly* (φ, φ)*-amenable Banach algebras is larger than that of* φ*-amenable Banach algebras.*

We need to recall the following remark for give the next example:

Remark 3.4. *Let* A *be a Banach algebra and* X *be a Banach* A*-module. Then* $A \oplus_1 X$*, the* l^1*-direct sum of* A *and* X *becomes a Banach algebra when equipped with the algebra product*

$$(a, x) \cdot (b, y) = (ab, a \cdot y + x \cdot b) \qquad (a, b \in A, \quad x, y \in X).$$

This Banach algebra is called module extension Banach algebras of A *and* X *(see [24]). Let* G *be a locally compact group,* $A = L^1(G)$ *and* $X = L^1(G)^* (= L^\infty(G))$*. Then by Proposition 5.1 of [24],* $L^1(G) \oplus_1 L^1(G)^*$ *is not weakly amenable. Let* $(e_\alpha)_\alpha$ *be a bounded approximate identity for* $L^1(G)$*, then it is easy to check that* $(e_\alpha, 0)_\alpha$ *is a bounded approximate identity for* $L^1(G) \oplus_1 L^1(G)^*$.

The following example give a non-weakly amenable Banach algebra which is weakly (φ, φ)*-amenable.*

Example 3.5. *Let* A *be a non-weakly amenable with a bounded approximate identity (for example let* $A = L^1(G) \oplus_1 L^1(G)^*$*). Then by Corollary 2.2 of [5],* $A^\#$ *(the unitization of* A*) is not weakly amenable. Define* $\varphi: A^\# \longrightarrow \mathbb{C}$ *by* $\varphi(a + \lambda) = \lambda \ (a \in A, \lambda \in \mathbb{C})$. *Clearly,* $\varphi \in \Delta(A^\#)$. *For every continuous* φ*-derivation* $D: A^\# \longrightarrow (A^\#_{(\varphi,\varphi)})^*$ *and* $a, b \in A$, $\lambda_1, \lambda_2 \in \mathbb{C}$ *we have,*

$$D\big((a + \lambda_1)(b + \lambda_2)\big) = \varphi(a + \lambda_1)D(b + \lambda_2) + \varphi(b + \lambda_2)D(a + \lambda_1)$$
$$= \lambda_1 D(b + \lambda_2) + \lambda_2 D(a + \lambda_1).$$

Let $(e_\alpha)_\alpha$ *be a bounded approximate identity for* A*. Therefore,*

$$D(a + 0) = \lim_\alpha D(a\, e_\alpha + 0) = \lim_\alpha D((a + 0)(e_\alpha + 0)) = 0 \ \ (a \in A).$$

Also,

$$D(0 + \lambda\ ) = D\big((0 + \lambda\ )(0 + 1)\big) = 2D(0 + \lambda\ ) \ \ (\lambda \in \mathbb{C}).$$

Thus $D(0 + \lambda) = 0$. *So*

$$D(a + \lambda) = D(a + 0) + D(0 + \lambda) = 0.$$

Therefore $A^\#$ *is weakly* (φ, φ)*-amenable.*

Remark 3.6. *We remind the reader that a commutative Banach algebra* A *is weakly amenable if and only if every continuous derivation* $D: A \longrightarrow X$*, for every symmetric Banach* A*-bimodule* X*, is inner [1].*

Proposition 3.7. *Let* A *be a Banach algebra and* $\varphi \in \Delta(A)$*. Then* A *is weakly* (φ, φ)*-amenable if and only if for every Banach space* X*, any continuous derivation* $D: A \longrightarrow X_{(\varphi,\varphi)}$ *is zero.*

Proof. *Suppose that* A *is weakly* (φ, φ)*-amenable and* $D: A \longrightarrow X_{(\varphi,\varphi)}$ *is a continuous derivation for some Banach space* X*. We show that* $D = 0$*. Assume towards a contradiction that* $D \neq 0$*. Thus there is* $a_0 \in A$ *and* $x^* \in X^*$ *such that* $\langle D(a_0), x^* \rangle \neq 0$*. Let* $T_x: A \longrightarrow \mathbb{C}$ *be defined by* $T_x(a) = \varphi(a)\langle x, x^* \rangle \ (a \in A)$*. Clearly,* $T_x \in A^*$*. Define* $T: X \longrightarrow A^*$ *by* $T(x) = T_x$ *and* $\widetilde{D}: A \longrightarrow (A_{(\varphi,\varphi)})^*$ *by* $\widetilde{D}(a) = T \circ D(a)$*. Obviously,* $\widetilde{D}$ *is a continuous derivation. From the fact that* A *is weakly* (φ, φ)*-amenable, it follows that* $\widetilde{D} = 0$*. Choose* $a' \in A$ *such that* $\varphi(a') = 1$*. Thus*

$$\langle D(a_0), x^* \rangle = \ \varphi(a')\langle D(a_0), x^* \rangle = \ \langle a', T_{D(a_0)} \rangle = \langle a', T \circ D(a_0) \rangle = \langle a', \widetilde{D}(a_0) \rangle = 0.$$

That is, $\langle D(a_0), x^* \rangle = 0$*. This contradicts the fact that* $\langle D(a_0), x^* \rangle \neq 0$*. Therefore* $D = 0$.
The converse is clear. □

The following example give a weakly amenable Banach algebra which is not weakly Δ*-amenable.*

Example 3.8. *Let* A *be a non-commutative Banach algebra,* X *be a Banach space and* $\varphi \in \Delta(A)$ *such that* $A \oplus X_{(\varphi,\varphi)}$ *is weaky amenable Banach algebra (for example let* $X = A$ *and* A *be a weakly amenable Banach algebra with a bounded approximate identity such that* $\mathrm{span}\{ab - ba: a, b \in A\}$ *is dense in* A*, see Theorem 5.4 of [20]). We show that* $A \oplus X_{(\varphi,\varphi)}$ *is not weakly* $((\varphi, 0), (\varphi, 0))$*-amenable. To this end , define*

$$D: A \oplus X_{(\varphi,\varphi)} \longrightarrow (X_{((\varphi,0),(\varphi,0))})^*$$

by $D(a,x) = x \ \ (a \in A, x \in X)$. *One can easily prove that* D *is a non zero continuous derivation. Thus by Proposition 3.7,* $A \oplus X_{(\varphi,\varphi)}$ *is not weakly* $((\varphi,0),(\varphi,0))$*-amenable.*

In Examples 3.5 and 3.8, we have shown that the weak Δ*-amenablity and weak amenablity are different.*

Example 3.9. *Let* A *be a commutative weakly amenable Banach algebra and* $\varphi \in \Delta(A)$. *By using Remark 3.6 and Proposition 3.7, one can easiliy show that* A *is weakly* (φ,φ)*-amenable. The converse is not true.*
To see this, let $A = L^1(G) \bigoplus_1 L^1(G)^*$, *where* G *is an abelian locally compact group. It follows from the Example 3.5 that* $A^{\#}$ *is a commutative weakly* (φ,φ)*-amenable Banach algebra but is not weakly amenable. So the class of weakly* (φ,φ)*-amenable Banach algebras is larger than that of weakly amenable commutative Banach algebras.*

Recall that a point derivation d *at a character* φ *of an algebra* A *is a linear functional* $d: A \longrightarrow \mathbb{C}$ *such that*

$$d(aa') = \varphi(a)d(a') + \varphi(a')d(a) \ \ (a, a' \in A).$$

In the following two examples we show that the weakly (φ,φ)*-amenability and weak* φ*-amenability are different.*

Example 3.10. *Let* A *be a Banach algebra with a bounded left approximate identity. Suppose that* A *admits a non-zero continuous poit derivation at* φ. *Then by Proposition 3.7,* A *is not weakly* (φ,φ)*-amenable but Theorem 2.2 of [18], implies that* A *is weakly* φ*-amenable.*

Example 3.11. *Consider the Banach algebra* $A = l^2(\mathbb{N})$ *of all sequences of complex numbers* $a := (a(n))$, *under pointwise operations, with norm*

$$\|a\|_2 = (\sum_{n=1}^{\infty} |a(n)|^2)^{\frac{1}{2}} < \infty.$$

Let $D: A \longrightarrow (A_{(0,0)})^*$ *be a continuous derivation. Then* $D(ab) = 0 \ (a, b \in A)$. *Since* A *has a left approximate identity, it follows that* $D = 0$. *Therefore* A *is not weakly* $(0,0)$*-amenable, but* A *is not weakly* 0*-amenable (see Example 2.3 of [18]).*

Proposition 3.12. *Let* A *and* B *be banach algebras and let* $h: A \longrightarrow B$ *be a continuous homomorphism with dense range. If* $\psi \in \Delta(B)$ *and* A *is weakly* $(\psi \circ h, \psi \circ h)$*-amenable, then* B *is weakly* (ψ,ψ)*-amenable.*
Proof. *Let* $D: B \longrightarrow (B_{(\psi,\psi)})^*$ *be a continuous derivation. Define* $\widetilde{D}: A \longrightarrow (A_{(\psi\circ h,\psi\circ h)})^*$ *by*

$$\widetilde{D}(a) = h^* \circ D \circ h(a) \quad (a \in A).$$

We can easily check that $\widetilde{D}$ *is a continuous derivation. From the weak* $(\psi \circ h, \psi \circ h)$*-amenability of* A, *it follows that* $\widetilde{D} = 0$. *By density of range of* h *and continuity of* D, *we conclude that* $D = 0$. *So* B *is weakly* (ψ,ψ)*-amenable.* □

Corollary 3.13. *Let* A *and* B *be banach algebras and let* $h: A \longrightarrow B$ *be a continuous homomorphism with dense range. If* A *is weakly* Δ*-amenable, then so is* B.

Lemma 3.14. *Let* A *be a Banach algebra and* I *be a closed ideal of* A. $\varphi \in \Delta(A)$ *such that* $\varphi|_I \neq 0$. *Let* $D: I \longrightarrow X_{(\varphi,\varphi)}$ *be a continuous derivation for some Banach space* X. *Then there is a bilinear map* $\overline{D}: I \times A \longrightarrow X_{(\varphi,\varphi)}$ *satisfying:*
(i) $\overline{D}(x,0)$ *extends* $\varphi|_I(x)D(.)$ *for every* $x \in I$;
(ii) *For every* $x \in I^2$, $\overline{D}(x,0)$ *is a continuous derivation.*
Proof. *Define* $\overline{D}: I \times A \longrightarrow X_{(\varphi,\varphi)}$ *by*

$$\overline{D}(x,a) = D(xa) - \varphi(a)D(x).$$

From the fact that D *is a derivation, it follows that*

$$\overline{D}(x,y) = \varphi|_I(x)D(y) \ \ (x, y \in I),$$

and so (i) *holds.*
(ii) *Clearly,* $\overline{D}(x,0)$ $(x \in I^2)$ *is continuous. For every* $x, y \in I$ *and* $a, b \in A$, *we have*

$$\overline{D}(xy,ab) = D(xyab) - \varphi(ab)D(xy) = D(yabx) - \varphi(ab)D(xy)$$

$$= D(abxy) - \varphi(ab)D(xy) = \varphi|_I(y)D(abx) - \varphi|_I(aby)D(x)$$
$$= \varphi|_I(y)D(abx) - \varphi(a)D(xyb) + \varphi|_I(ax)D(yb) = \varphi|_I(y)D(abx) - \varphi(a)D(bxy) + \varphi|_I(ax)D(yb)$$
$$= \varphi|_I(y)D(abx) - \varphi|_I(abx)D(y) + \varphi|_I(ay)D(bx) + \varphi|_I(ax)D(yb)$$
$$= \varphi|_I(y)D(abx) - \varphi|_I(abx)D(y) - D(yabx) + \varphi|_I(bx)D(ya) + \varphi|_I(ax)D(yb)$$
$$= \varphi|_I(bx)D(ya) + \varphi|_I(ax)D(yb) - 2\varphi|_I(bax)D(y)$$
$$= \varphi(b)\big(\varphi|_I(x)D(ya) - \varphi|_I(ax)D(y)\big) + \varphi(a)\big(\varphi|_I(x)D(yb) - \varphi|_I(bx)D(y)\big)$$
$$= \varphi(b)\big(D(xya) - \varphi(a)D(xy)\big) + \varphi(a)\big(D(xyb) - \varphi(b)D(xy)\big)$$
$$= \bar{D}(xy,a)\cdot b + a\cdot\bar{D}(xy,b).$$

So $\bar{D}(x,0)$ *is a continuous derivation for every* $x \in I^2$. □

The following two propositions are needed for proof of the main result in the next section.

Proposition 3.15. *Let* A *be a Banach algebra and* I *be a closed ideal of* A. $\varphi \in \Delta(A)$ *such that* $\varphi|_I \neq 0$. *If* A *is weakly* (φ,φ)*-amenability, then* I *is weakly* $(\varphi|_I,\varphi|_I)$*-amenability.*

Proof. *Suppose that* A *is weakly* (φ,φ)*-amenable and let* $D: I \longrightarrow (I_{(\varphi|_I,\varphi|_I)})^*$ *be a continuous derivation. Define a linear map* $T: I^* \longrightarrow B(I,I^*)$ *by*

$$T(f)(i) = i\cdot f \quad (i \in I, f \in I^*),$$

Where $B(I,I^*)$ *be the space of all bounded linear maps from* I *to* I^*. *Then* $T \circ D$ *is a continuous derivation from* I *into* $B(I,I^*)_{(\varphi,\varphi)}$. *Let* $\bar{D}$ *be the corresponding bilinear map from* $I \times A$ *into* $B(I,I^*)_{(\varphi,\varphi)}$. *By Lemma 3.14,* $\bar{D}(x,0)$ *is a derivation from* A *into* $B(I,I^*)_{(\varphi,\varphi)}$ *for all* $x \in I^2$. *Now Proposition 3.7, implies that* $\bar{D}(x,0) = 0$. *Again Lemma 3.14, yield that* $\varphi(I^2)D(I) = \{0\}$. *Choose* $i_0 \in I$ *such that* $\varphi|_I(i_0) = 1$. *Then for every* $i \in I$, $D(i) = \varphi(i_0 i_0)D(i) = 0$, *that is* $D = 0$. *Consequently* I *is a weakly* $(\varphi|_I,\varphi|_I)$*-amenable Banach algebra.* □

Let A *be a Banach algebra and* $\varphi \in \Delta(A)$. *Define* $\varphi^{\#}: A^{\#} \longrightarrow \mathbb{C}$ *by*

$$\varphi^{\#}(a+\lambda) = \varphi(a) + \lambda \quad (\lambda \in \mathbb{C}, a \in A).$$

Then $\varphi^{\#} \in \Delta(A^{\#})$, *and* $\varphi^{\#}|_A = \varphi$.

Proposition 3.16. *Let* A *be a Banach algebra and* $\varphi \in \Delta(A)$. *Then* $A^{\#}$ *is a weakly* $(\varphi^{\#},\varphi^{\#})$*-amenable if and only if* A *is weakly* (φ,φ)*-amenable.*

Proof. *Assume that* $A^{\#}$ *is a weakly* $(\varphi^{\#},\varphi^{\#})$*-amenable. Then Proposition 3.15 yield that* A *is weakly* (φ,φ)*-amenable.*

Conversely, suppose that A *is weakly* (φ,φ)*-amenable. By Proposition 3.7, it suffices to show that for every Banach space* X, *every continuous derivation from* $A^{\#}$ *into* $X_{(\varphi^{\#},\varphi^{\#})}$ *is zero. Thus, let* $D: A^{\#} \longrightarrow X_{(\varphi^{\#},\varphi^{\#})}$ *be a continuous derivation. Then* $D|_A$ *is a continuous derivation and by Proposition 3.7, we conclude that* $D|_A = 0$. *Also for every* $\lambda \in \mathbb{C}$,

$$D(0+\lambda) = D((0+\lambda)(0+1)) = \lambda D(0+1) + D(0+\lambda).$$

That is $D(0+\lambda) = 0$. *Consequently,*

$$D(a+\lambda) = D(a+0) + D(0+\lambda) = 0,$$

for all $a \in A$ *and* $\lambda \in \mathbb{C}$. *So* $D = 0$. *This completes the proof.* □

4. *Weak* (φ,φ)*-amenability of projective tensor product* $\mathbf{A}\,\widehat{\otimes}\,\mathbf{B}$

It is well known that $A\,\widehat{\otimes}\,B$, *the projective tensor product of* A *and* B *is Banach algebra with respect to the canonical multiplication defined by*

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2 \otimes b_1 b_2) \qquad (a_1, a_2 \in A, b_1, b_2 \in B).$$

For $f \in A^*$ *and* $g \in B^*$, *let* $f \otimes g$ *denote the element of* $A\,\widehat{\otimes}\,B^*$ *satisfying* $f \otimes g(a \otimes b) = f(a)g(a)$ *for all* $a \in A$ *and* $b \in B$. *Then, with this notion,*

$$\Delta(A\,\widehat{\otimes}\,B) = \{\varphi \otimes \psi : \varphi \in \Delta(A), \psi \in \Delta(B)\}.$$

Gronbaek in [6], proved that if A *and* B *are commutative weakly amenable Banach algebras, then* $A\,\widehat{\otimes}\,B$ *is weakly amenable and Yazdanpanah in [23], showed that the converse is also valid. Moreover authors in [11], showed that if* A *is* φ*-commutative and* B *is* ψ*-commutative, where* $\varphi \in Hom(A)$ *and* $\psi \in Hom(B)$, *then*

$A \widehat{\otimes} B$ is $\varphi\otimes\psi$-weakly amenable if and only if A is φ-weakly amenable and B is ψ-weakly amenable. In the following theorem, we remove commutative condition and prove similar results for weakly $(\varphi\otimes\psi, \varphi\otimes\psi)$-amenability of $A \widehat{\otimes} B$, where $\varphi \in \Delta(A)$ and $\psi \in \Delta(B)$.

Theorem 4.1. *Let A and B be Banach algebras and let $\varphi \in \Delta(A)$ and $\psi \in \Delta(B)$. Then $A \widehat{\otimes} B$ is weakly $(\varphi\otimes\psi, \varphi\otimes\psi)$-amenable if and only if A is weakly φ- amenable and B is weakly ψ-amenable.*

Proof. *Suppose that $A \widehat{\otimes} B$ is weakly $(\varphi\otimes\psi, \varphi\otimes\psi)$-amenable. Let $D: A \longrightarrow (A_{(\varphi,\varphi)})^*$ be a continuous derivation. Define $\overline{D}: A \widehat{\otimes} B \longrightarrow (A \widehat{\otimes} B_{(\varphi\otimes\psi,\varphi\otimes\psi)})^*$ by*

$$\langle a'\otimes \mathrm{b}', \overline{D}(a\otimes \mathrm{b})\rangle = \psi(b'b)\langle a', D(a)\rangle \quad (a, a' \in A, b, b' \in B).$$

Clearly, $\overline{D}$ is continuous and for every a_1, a_2, $a' \in A$ and $b_1, b_2, b' \in B$, we have

$$\begin{aligned}\langle a'\otimes \mathrm{b}', \overline{D}((a_1\otimes b_1)(a_2\otimes b_2))\rangle &= \psi(b'b_1b_2)\langle a', D(a_1a_2)\rangle\\ &= \varphi(a_1)\psi(b'b_1b_2)\langle a', D(a_2)\rangle + \varphi(a_2)\psi(b'b_1b_2)\langle a', D(a_1)\rangle\\ &= \langle a'\otimes \mathrm{b}', \varphi\otimes\psi(a_1\otimes b_1)\overline{D}(a_2\otimes b_2)\rangle + \langle a'\otimes \mathrm{b}', \varphi\otimes\psi(a_2\otimes b_2)\overline{D}(a_1\otimes b_1)\rangle\\ &= \langle a'\otimes \mathrm{b}', (a_1\otimes b_1)\cdot\overline{D}(a_2\otimes b_2)\rangle + \langle a'\otimes \mathrm{b}', \overline{D}(a_1\otimes b_1)\cdot(a_2\otimes b_2)\rangle\end{aligned}$$

Thus $\overline{D}$ is a derivation. Now weak $(\varphi\otimes\psi, \varphi\otimes\psi)$-amenability of $A \widehat{\otimes} B$, implies that $\overline{D} = 0$. Choose $b_0 \in B$ such that $\psi(b_0) = 1$. Hence, for every $a, a' \in A$, we have

$$\langle a', D(a)\rangle = \psi(b_0b_0)\langle a', D(a)\rangle = \langle a'\otimes b_0, \overline{D}(a\otimes b_0)\rangle = 0.$$

That is $D = 0$. So A is weakly φ-amenable. Similar arguments show that B is weakly ψ-amenable.

Conversely, suppose that A is weakly φ-amenable and B is weakly ψ-amenable. Then by Proposition 3.16, $A^\#$ is weakly $(\varphi^\#, \varphi^\#)$-amenable and $B^\#$ is weakly $(\psi^\#, \psi^\#)$-amenable. Let X be a Banach space and $D^\#: A^\# \widehat{\otimes} B^\# \longrightarrow X_{(\varphi^\#\otimes\psi^\#, \varphi^\#\otimes\psi^\#)}$ be a continuous derivation. Take $D = D^\#|_{A^\#\widehat{\otimes}1_{B^\#}}$, and define

$$\mathrm{D}_{A^\#}: A^\# \longrightarrow X_{(\varphi^\#, \varphi^\#)}, \quad (a + \lambda) \mapsto D((a + \lambda)\otimes 1_{B^\#}).$$

It is easily seen that $\mathrm{D}_{A^\#}$ is a continuous derivation. Since $A^\#$ is weakly $(\varphi^\#, \varphi^\#)$-amenable, Proposition 3.7, implies that $\mathrm{D}_{A^\#} = 0$. So $D = D^\#|_{A^\#\widehat{\otimes}1_{B^\#}} = 0$. Similarly, if we take $D = D^\#|_{1_{A^\#}\widehat{\otimes}B^\#}$, one can show that $D = 0$. Therefore, from the fact that

$$A^\# \widehat{\otimes} B^\# = \overline{((A^\# \widehat{\otimes} 1_{B^\#})(1_{A^\#} \widehat{\otimes} B^\#))}$$

it follows that $D^\# = 0$. Thus, $A^\# \widehat{\otimes} B^\#$ is weakly $(\varphi^\#\otimes\psi^\#, \varphi^\#\otimes\psi^\#)$-amenable. Since $A \widehat{\otimes} B$ is a closed ideal of $A^\# \widehat{\otimes} B^\#$, Proposition 3.15, yields that $A \widehat{\otimes} B$ is weakly $(\varphi\otimes\psi, \varphi\otimes\psi)$-amenable. □

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