Linear Fuzzy Singular Volterra Integral Equations by applying Horizontal Membership Function Approach

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ABSTRACT

In this paper, our goal is to find an exact solution for the Linear Fuzzy Singular Volterra Integral Equations (LFSVIEs) of the first type $\tilde{f}(x) = \oint_0^x \frac{1}{(x-t)^\alpha} \odot_{gr} \tilde{y}(t)\,dt$ *and the second type* $\tilde{y}(x) = \tilde{f}(x) \oplus_{gr} \oint_0^x \frac{1}{(x-t)^\alpha} \odot_{gr} \tilde{y}(t)\,dt$, *where* $0 < \alpha < 1$ *under granular approach. In the examples mentioned in this paper, we use the Laplace transform tools to find the exact solution.*

Keywords: Linear Fuzzy Singular Volterra Integral Equations (LFSVIEs), granular approach, Horizontal Membership Function (HMF).

1. INTRODUCTION

Fuzzy set theory, introduced by Zadeh [1], has become a powerful tool for modeling uncertainty and vagueness in real world problems. Over the past decades, fuzzy integral equations have found numerous applications in control theory, signal processing, population dynamics, and engineering systems. Among them, Linear Fuzzy Singular Volterra Integral Equations (LFSVIEs) with singular kernels of the form $\frac{1}{(x-t)^\alpha}$ *(where* $0 < \alpha < 1$*) are particularly important yet challenging to solve due to the singularity at* $t = x$.

Traditional approaches based on classical fuzzy arithmetic often suffer from the draw back of overestimation and unnatural uncertainty propagation as time evolves [2]. To address these limitations, the granular computing paradigm and the Horizontal Membership Function (HMF) approach, developed by Piegat and Landowski [3] and further advanced by Mazandarani et al. [4], have emerged as effective methodologies. The gran ular approach, combined with the Relative-Distance-Measure (RDM) variable, provides a more consistent and computationally reliable framework for fuzzy arithmetic operations.

Several researchers have contributed to the study of fuzzy Volterra integral equations. Salahshour and Allahviranloo [2] applied the fuzzy differential transform method, while Wazwaz [5] provided comprehensive analytical techniques for crisp linear and nonlinear integral equations. More recently, the stability of fuzzy linear dynamical systems under granular arithmetic has been investigated by Najaryian and Zhao [6].

Despite these advancements, the literature still lacks exact solutions for Linear Fuzzy Singular Volterra Integral Equations under the granular framework. The present paper aims to fill this gap by deriving exact solutions for both the first and second kinds of LFSVIEs using the Horizontal Membership Function(HMF) approach. By transforming the fuzzy equations into crisp parameterized forms and applying the Laplace transform method, we obtain closed form solutions expressed in terms of the Mittag-Leffler function. The effectiveness of the proposed method is illustrated through numerical examples.

The rest of the paper is organized as follows: Section 2 provides the necessary preliminaries on fuzzy numbers and granular calculus. The main results are presented in Section 3, followed by illustrative examples in Section 4. Finally, conclusions are discussed in Section 5.

2. Preliminaries

This section recalls fundamental concepts from fuzzy set theory and granular computing that are essential for the sebsequent analysis. Key notions include fuzzy numbers, level sets, the Horizontal Membership Function (HMF), granular arithmetic operations and the granular fuzzy Laplace transform.

Definition 2.1. *A type-1 fuzzy set, denoted by $\tilde{y}$ and mapping $\mathbb{R}$ to $[0.1]$, qualifies as a type-1 fuzzy number if it satisfies the following properties: it is normal, fuzzy convex, upper semi-continuous, and has a compact support.*

Here, we denote by $\mathbb{R}_F$ the space of type 1-fuzzy numbers.

Definition 2.2. *For a membership level μ where $0 < \mu \leq 1$, the -level set of a fuzzy number $\tilde{y}$ is defined as $[\tilde{y}]^{\mu} = \{x \in \mathbb{R}\,;\ \tilde{y}(x) \geq \mu\}$. The 0-level set, or support, is $[\tilde{y}]^{0} = \{x \in \mathbb{R}\,;\ \tilde{y}(x) > 0\}$. The 1-level set is termed the core. Each μ-level set is a closed interval expressed as $[\tilde{y}]^{\mu} = [\underline{\tilde{y}}^{\mu}.\overline{\tilde{y}}^{\mu}]$.*

Definition 2.3. *Consider a fuzzy function $\tilde{y} : [a.b] \subseteq \mathbb{R} \to \mathbb{R}_F$. Its horizontal member ship function (HMF) is a mapping $\tilde{y}_{gr} : [0.1] \times [0.1] \to [a.b]$, defined for a point x by $\tilde{y}_{gr}(\mu.\ \alpha_y) = x$. Here, $\mu \in [0.1]$ is the membership grade and $\alpha_y \in [0.1]$ is a Relative Distance-Measure (RDM) variable. This function is explicitly parameterized using the μ -cut $[\underline{\tilde{y}}^{\mu}.\overline{\tilde{y}}^{\mu}]$ as:*

$$\tilde{y}_{gr}(\mu.\ \alpha_y) = \underline{\tilde{y}}^{\mu} + \left(\overline{\tilde{y}}^{\mu} - \underline{\tilde{y}}^{\mu}\right)\ \alpha_y,$$

(refer to [3],[4])

Remark 2.4. *The symbol $H(\tilde{y}(x)) \equiv \tilde{y}_{gr}(x.\ \alpha_y.\mu)$ denotes the horizontal membership function of a fuzzy-valued function $\tilde{y}(x)$. The inverse operation, which recovers the μ-level sets from the HMF, is given by:*

$$\mathrm{H}^{-1}(\tilde{y}_{gr}(x.\ \alpha_y.\mu)) = [\tilde{y}(x)]^{\mu} = [\inf_{\gamma\geq\mu} \min_{\alpha_y} \tilde{y}_{gr}(x.\ \alpha_y.\gamma).\sup_{\gamma\geq\mu} \max_{\alpha_y} \tilde{y}_{gr}(x.\ \alpha_y.\gamma)],$$

(see [4])

Definition 2.5. *Let $\check{y}$ and $\tilde{z}$ be two fuzzy numbers. The granular metric D_{gr}on the space of fuzzy numbers $\mathbb{R}_F$ is defined as the function D_{gr}: $\mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^{+} \cup \{0\}$, given by:*

$$D_{gr}(\tilde{y}.\check{z}) = \sup_{\mu\in[0.1]} \max_{\alpha_y.\alpha_z\in[0.1]} |\tilde{y}(\mu.\alpha_y) - \tilde{z}(\mu.\alpha_z)|,$$

(refer to [4])

Definition 2.6. *Let $\tilde{x}$ and $\tilde{y}$ be type-1 fuzzy numbers with horizontal membership func tions $\tilde{x}_{gr}(\mu.\alpha_x)$ and $\tilde{y}_{gr}(\mu.\alpha_y)$ respectively. For any fundamental arithmetic operation $*_{gr}$ (addition, subtraction, multiplication, or division), the result $\tilde{z} = \tilde{x} *_{gr} \tilde{y}$ is a type-1 fuzzy number whose horizontal membership function is defined by the pointwise operation:*

$$\tilde{\mathrm{z}}_{\mathrm{gr}}(\mu.\alpha_{\mathrm{z}}) = \tilde{\mathrm{x}}_{gr}(\mu.\alpha_{\mathrm{x}})\tilde{\mathrm{y}}_{gr}(\mu.\alpha_{\mathrm{y}}),$$

A necessary condition for division is that zero is not contained within the support of the divisor $\tilde{y}$.*(see [4])*

Definition 2.7. *The space comprising fuzzy functions* $\tilde{f}: [a.b] \subseteq \mathbb{R} \to \mathbb{R}_F$ *that are continuous on the interval* D_{gr} *with respect to the granular metric is denoted by* $C_{gr}([a.b])$.

Definition 2.8. *If* $\tilde{f} \in C_{gr}([a.b])$, *its granular integral is defined by*

$$\mathrm{gr}\left(\oint_{\mathrm{a}}^{\mathrm{b}} \tilde{\mathrm{f}}(\mathrm{x})\, \mathrm{dx}\right) = \int_{\mathrm{a}}^{\mathrm{b}} \tilde{\mathrm{f}}_{\mathrm{gr}}\,(\mathrm{x}.\mu.\alpha_{\mathrm{f}})\, \mathrm{dx}.$$

where $\alpha_f \in [0.1]$, *(see [4])*

Definition 2.9. *The granular fuzzy Laplace transform of a continuous fuzzy function* $\tilde{f}(x)$ *is defined as:*

$$\mathrm{L}_{\mathrm{gr}}[\tilde{\mathrm{f}}(\mathrm{x})] = \oint_0^\infty \tilde{\mathrm{f}}\,(\mathrm{x}) \odot_{\mathrm{gr}} \mathrm{e}^{-\mathrm{sx}}\mathrm{dx}$$

provided that the granular improper fuzzy integral of $\tilde{f}(x) \odot_{gr} e^{-sx}$ *exists on the interval* $[0.\infty)$.*(see [6])*

3. Main results

In this section, consider the following LFSVIE of the first type

$$\tilde{\mathrm{f}}(\mathrm{x}) = \oint_0^{\mathrm{x}} \frac{1}{(\mathrm{x}-\mathrm{t})^{\alpha}} \odot_{\mathrm{gr}} \tilde{\mathrm{y}}(\mathrm{t})\, \mathrm{dt}. \quad \mathrm{x} \in [0.\mathrm{T}] \tag{1}$$

And the second type

$$\tilde{\mathrm{y}}(\mathrm{x}) = \tilde{f}(x) \oplus_{\mathrm{gr}} \oint_0^{x} \frac{1}{(x-t)^{\alpha}} \odot_{gr} \tilde{y}(t)\, dt. \quad x \in [0.\mathrm{T}] \tag{2}$$

where $0 < \alpha < 1, \tilde{f} \in C_{gr}([0.T])$ *and* $k(x.t) \in C([0.T] \times [0.T])$. *The procedure for solving the Integral equations (1) and (2) is as follows: First, we take the granular of both sides of the fuzzy integral equation or compute its HMF according to the* ***Definition 2.3***. *Then, fuzzy integral equations (1) and (2) are transformed into a crisp integral equations, and here, we find its exact solution using the Laplace transform. Finally, using* ***Remark 2.4***, *we obtain the* μ*-level set of the solution.*

4. Examples

This section demonstrates the practical application of the proposed granular approach through two illustrative examples, one for a second type and one for a first type LFSVIE, highlighting the solution procedure and the resulting exact fuzzy solutions.

Example 4.1. We consider the following (LFSVIE) of the second type

$$\tilde{y}(\mathrm{x}) = x \odot_{gr} [\mu.2-\mu] \oplus_{gr} x^{\frac{5}{3}} \odot_{gr} \left[-1+\frac{\mu}{10}.-\frac{9}{10}\mu\right] \oplus_{gr} \oint_0^x \frac{1}{(x-t)^{\frac{1}{3}}} \odot_{gr} \tilde{y}(t)\, dt, \tag{3}$$

First of all, for each fixed $\mu.\alpha_1.\alpha_2 \in [0.1]$, we transform the fuzzy integral equation (3) into a crisp integral equation using the HMF, then we obtain

$$\tilde{y}_{gr}(\mathrm{x}) = \mathrm{x}(\mu + 2(1-\mu)\alpha_1) + x^{\frac{5}{3}}\left(-1+\frac{\mu}{10} + (1-\mu)\alpha_2\right) + \int_0^{\mathrm{x}} \frac{\tilde{y}_{gr}(\mathrm{t})}{(\mathrm{x}-\mathrm{t})^{\frac{1}{3}}}\,\mathrm{dt},$$

Then, according to **Definition 2.9**, we use the Laplace transform and we have

$$\mathrm{L}\{\tilde{y}_{gr}(\mathrm{x})\} = \frac{(\mu + 2(1-\mu)\alpha_1)}{s^{\frac{4}{3}}(s^{\frac{2}{3}} - \Gamma\left(\frac{2}{3}\right))} + \frac{\Gamma(\frac{8}{3})(-1+\frac{\mu}{10} + (1-\mu)\alpha_2)}{s^2(s^{\frac{2}{3}} - \Gamma\left(\frac{2}{3}\right))},$$

($E_{\alpha.\beta}(x).\boldsymbol{\Gamma}(\boldsymbol{x})$ are the Mittag-Leffler function and the Euler gamma function, repectively.) that $\tilde{Y}_{gr}(s) = L\{\tilde{y}_{gr}(\mathrm{x})\}$. By taking the inverse Laplace transform from both sides

$$\tilde{y}_{gr}(\mathrm{x}) = (\mu + 2(1-\mu)\alpha_1)\mathrm{x}E_{\frac{2}{3}.2}\left(\Gamma\left(\frac{2}{3}\right)\mathrm{x}^{\frac{2}{3}}\right) + \frac{\Gamma\left(\frac{8}{3}\right)\left(-1+\frac{\mu}{10}+(1-\mu)\alpha_2\right)}{\Gamma\left(\frac{2}{3}\right)}\mathrm{x}\left(E_{\frac{2}{3}.2}\left(\Gamma\left(\frac{2}{3}\right)x^{\frac{2}{3}}\right)-1\right),$$

Now, using **Remark 2.4**, we arrive at the following exact solution

$$[\tilde{y}]^{\mu} = \left[\frac{x}{9}\left(10(\mu-1)\left(E_{\frac{2}{3}.2}\left(\Gamma\left(\frac{2}{3}\right)x^{\frac{2}{3}}\right)-1\right)+10-\mu\right).x\left(2(1-\mu)E_{\frac{2}{3}.2}\left(\Gamma\left(\frac{2}{3}\right)x^{\frac{2}{3}}\right)+\mu\right)\right],$$

Example 4.2. Let us examine the following (LFSVIE) of the first type

$$\left[\frac{23}{5}+\frac{\mu}{5}, 5-\frac{\mu}{5}\right] \odot_{\mathrm{gr}} \mathrm{x}^{\frac{5}{6}} = \oint_0^{\mathrm{x}} \frac{1}{(\mathrm{x}-\mathrm{t})^{\frac{1}{6}}} \odot_{\mathrm{gr}} \tilde{\mathrm{y}}(\mathrm{t})\,\mathrm{dt}. \tag{4}$$

Applying the HMF approach, we transform the fuzzy equation into a crisp parameterized form. For fixed $\mu.\alpha \in [0.1]$, the equation becomes

$$\left(\frac{23}{5}+\frac{\mu}{5}+\frac{2}{5}(1-\mu)\alpha\right)\mathrm{x}^{\frac{5}{6}} = \int_0^{\mathrm{x}} \frac{\tilde{y}_{gr}(\mathrm{t})}{(\mathrm{x}-\mathrm{t})^{\frac{1}{6}}}\,\mathrm{dt},$$

Taking the Laplace transform of both sides and simplifies to

$$\mathrm{L}\{\tilde{y}_{gr}(\mathrm{x})\} = \frac{5}{6s}\left(\frac{23}{5}+\frac{\mu}{5}+\frac{2}{5}(1-\mu)\alpha\right).$$

The inverse Laplace transform gives the solution in the granular form

$$L\{\tilde{y}_{gr}(\mathrm{x})\} = \frac{5}{6}\left(\frac{23}{5}+\frac{\mu}{5}+\frac{2}{5}(1-\mu)\alpha\right).$$

Finally, applying the inverse HMF transformation, we obtain the fuzzy solution in terms of its μ-level sets

$$[\tilde{y}]^{\mu} = \left[\frac{23}{6} + \frac{\mu}{6}, \frac{25}{6} - \frac{\mu}{6}\right],$$

5. Conclusion

In this paper, we successfully developed a methodology for obtaining exact solutions of Linear Fuzzy Singular Volterra Integral Equations (LFSVIEs) of the first and second types using the granular approach and Horizontal Membership Function (HMF). By transforming the original fuzzy equations into parameterized crisp integral equations and applying the Laplace transform technique, we derived closed form solutions in terms of the Mittag-Leffler function.

The obtained results demonstrate that the granular HMF approach provides a more reliable and computationally efficient framework compared to classical fuzzy methods. The solutions maintain the proper fuzzy properties and significantly reduce the overestimation phenomenon typically associated with standard fuzzy arithmetic operations.

REFERENCES

[1] L. A. Zadeh, "Fuzzy sets," *Information and Control,* vol. 8, no. 3, pp. 338–353, 1965/06/01/ 1965, doi: https://doi.org/10.1016/S0019-9958(65)90241-X.

[2] S. Salahshour and T. Allahviranloo, "Application of fuzzy differential transform method for solving fuzzy Volterra integral equations," *Applied Mathematical Modelling,* vol. 37, no. 3, pp. 1016–1027, 2013/02/01/ 2013, doi: https://doi.org/10.1016/j.apm.2012.03.031.

[3] A. Piegat and M. Landowski, "Horizontal Membership Function and Examples of its Applications," *International Journal of Fuzzy Systems,* vol. 17, no. 1, pp. 22–30, 2015/03/01 2015, doi: 10.1007/s40815-015-0013-8.

[4] M. Mazandarani, N. Pariz, and A. V. Kamyad, "Granular Differentiability of Fuzzy-Number-Valued Functions," *IEEE Transactions on Fuzzy Systems,* vol. 26, no. 1, pp. 310–323, 2018, doi: 10.1109/TFUZZ.2017.2659731.

[5] A.-M. Wazwaz, *Linear and Nonlinear Integral Equations Methods and Applications*, 1 ed. Springer Berlin, Heidelberg, 2011, p. 639.

[6] M. Najariyan and Y. Zhao, "On the stability of fuzzy linear dynamical systems," *Journal of the Franklin Institute,* vol. 357, no. 9, pp. 5502–5522, 2020/06/01/ 2020, doi: https://doi.org/10.1016/j.jfranklin.2020.02.023.